A critique of a Post Keynesian model of hoarding, and an alternative model

TROND ANDRESEN[a]

Abstract

The concept of a "propensity to hoard" is frequently used by Post Keynesians and Circuitists in time-discrete models of the macroeconomy, to account for how households manage their flow of savings. This concept is argued to be erroneous – and continuous circuit models are better than discrete for a clear understanding of this. The phenomenon of time dispersion of circulating money is discussed – also impulse functions and first-order differential equations as building blocks in a network. Finally, these components and concepts are used to assemble and simulate a circuit model with debt. Even at high interest and savings rates the system evolves without ending in debt-induced crisis.

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a. The Norwegian University of Science and Technology
Faculty of Information Technology, Mathematics and Electrical Engineering
Department of Engineering Cybernetics
N-7491 Trondheim, NORWAY
Phone: +47 7359 4358, Fax: +47 7359 4399 (work)
Phone: +47 7353 0823 (home), +47 9189 7045 (mobile)

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1. Introduction

There is an approach to macroeconomic modeling that portrays the economy essentially as a circulatory flow of money: households to firms to households, with the role for banks as sources and sinks for money flows. The fundamental issue is to examine what is needed for this circulatory flow to be maintained, and what sort of events or mechanisms can lead to a damaging decrease in circulation and ensuing crisis. This is central to Post Keynesians and Circuitists – representative proponents are Lavoie (1992) and Graziani (1996). A necessary element of this approach is that since money obviously takes time to complete a round (else money velocity would be infinite), there must be some way of allowing for delays or money storage somewhere in the circuit. A view of money as a fluid-like entity which flows and is lagged within sectors can be traced back to A. W. Phillips' "hydraulic" macroeconomic model (e.g. Phillips 1954). His portrayal of the firm sector in this paper corresponds well (at least as an approximation for small changes in flows and levels) to the dynamics of an open vessel with a fixed aperture at the bottom: A sudden increase in the incoming flow (demand) will gradually increase the level of fluid (read: money held by firms), which again leads to an increased outflow (income).

The concept of a "propensity to hoard" (or the complementary concept, the "propensity to buy securities from firms") is frequently used by Post Keynesians and Circuitists to signify how households manage their flow of savings in the circuit. This paper

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intends – in the spirit of improving the circuit approach which the author sympathises with – to show that this concept is flawed: it implies unbounded accumulation of money with households and persistently decreasing money velocity – even in an otherwise equilibrium case when all other variables are constant. The main reason for this error is a common modeling assumption in the PK/C literature about hoarding which – without recognising this – implies *that money is permanently taken out of circulation*. The prime purpose of this paper is to undertake criticism of this. It will also be argued that there are other serious errors that result from solely verbal analysis of the economic circuit. The arguments will be given using tools from linear systems theory.

The paper is organised as follows: In section 2, the standard PK/C model is presented and criticised. In section 3 a better model is introduced, and the paper transits to a continuous time platform for the remaining analysis. In section 4 a more comprehensive alternative model is presented, simulated and discussed. Conclusions are drawn in section 5.

2. The basic model

One fairly recent paper, Fontana (2000) will be used as a point of departure, since his paper gives an overview of PK/C treatment of the monetary circuit. (A similar analysis is found in for instance Lavoie (1992, 152 - 157) The set of equations on pp. 34 - 37 in Fontana's paper contains what will from now on be for convenience called a PK/C model of the macroeconomic monetary circulatory system. This is the model which is to be examined. We will from now on portray this, and other, models in block diagram[1] form. This will be seen to be useful for our analysis. Consider figure 1:

----- Figure 1 about here -----

This is a block diagram of Fontana's equation-based model. The notation is the same.

The block diagram is equivalent to the set of equations (1) – (3) presented below. The upper *summation circle* in the diagram signifies the equation

$$I = s_F\Pi + m \cdot s_{WE}Y_{WE}, \text{ where } s_F\Pi = s_F\pi Y \quad (1)$$

Profits Π are a share π of aggregate income Y. Aggregate investment is funded by firms' retained earnings (the share s_F of Π not used for consumption by capitalists) plus the proceeds of securities sold to wage earners. The coefficient s_{WE} is wage earners' propensity to save. The parameter *m* was proposed by Davidson (1972, 272), as the "marginal propensity to buy placements out of household savings".

If banks are extending net loans ΔM^S in the period, an option suggested by the dotted arrow, eq. (1) is modified to $I = s_F\Pi + m \cdot s_{WE}Y_{WE} + \Delta M^S$ (2)

The two other summation circles signify

$$C + I = C_{WE} + C_F + I \text{ where } C_{WE} = (1 - s_{WE})Y_{WE} \text{ and } C_F = (1 - s_F)\pi Y \quad (3)$$

Here Y_{WE} and C_{WE} are wage earners' income and consumption respectively, C_F is capitalists' consumption.

At this stage of the presentation, we assume no net lending, $\Delta M^S = 0$ (which corresponds to removing the dotted arrow in the block diagram). Assume at this stage also that all income received by households is either consumed or used to buy securities, i.e. $m = 1$. We then have from equations (1) and (3), or from the block diagram:

1. A block diagram is equivalent to a set of equations. Explanation of block diagram symbolism:
   - Except for the two blocks for "wage earners" and "firms" shaded and outlined with bold lines, the remaining rectangular blocks contain constants, to be interpreted such that the variable exiting a constant block is the product of the constant in the block and the incoming variable.
   - C*ircles* imply summation: the variable exiting a circle is the sum of the incoming variables.
   - *Small filled dots* imply that the corresponding variable is employed in more than one relationship. We observe for instance that output *Y* is split into profits and wage earners' income. Note that coefficients after a dot sum to unity; money is shared, not created or destroyed when flows are split.

$$C + I = C_{WE} + C_F + I = (1 - s_{WE})Y_{WE} + (1 - s_F)\pi Y + s_{WE}Y_{WE} + s_F\Pi =$$
$$(1 - s_{WE})(1 - \pi)Y + (1 - s_F)\pi Y + s_{WE}(1 - \pi)Y + s_F\pi Y = \qquad (4)$$
$$(1 - \pi)Y + \pi Y = Y$$

In this case our theoretical monetary circuit may continue forever in an equilibrium state. This is more easily seen by inspecting the block diagram: Since $m = 1$, all money entering the wage earners' block exits from the same block and appears at the input of the firms block – thus we must have $C + I = Y$. Firms get back all they have paid to wage earners. The only possible explanation for the alternative, $C + I < Y$, must be a "sink" for money, i.e. money hoarding. This happens for $m < 1$. The argument is based on "one period" of the circuit: After an initial aggregate investment (financed through an initial bank loan), at the end of the "period" firms will get back the amount invested less the amount hoarded by wage earners, which is $s_{WE}(1 - m)Y_{WE}$. So, to ensure that firms receive as much as they have originally borrowed for the initial investment, they must be furnished with a new loan before the end of the period,

$$\Delta M^S = s_{WE}(1 - m)Y_{WE} \qquad (5)$$

[Fontana (2000,36)]. Then they are in the same position at the start of the next period as at the start of the first and the process may be repeated.

Now consider the modified block diagram in figure 2, which is identical to figure 1 except that the wage earners block is "opened up", to indicate the PK/C model of wage earners' hoarding:

----- Figure 2 about here -----

The analysis in Fontana's paper is based on just one (unspecified) time period. Let us introduce several successive periods, but not specify the actual length of a period (we will introduce "physical time" later on). We use the symbolism $Y[t], \Delta M^S[t]$, etc. to indicate the value of a *flow*, which is assumed constant between time instants $t,\ t + 1$. The money hoard $M_{WE}[t]$ is a *stock*, at the time instant *t*. Since $s_{WE}(1 - m)Y_{WE} = s_{WE}(1 - m)(1 - \pi)Y$

is hoarded in each period, we get, for $t = 1, 2, \ldots$:

$$\begin{aligned} Y[t] &= Y[t-1] - s_{WE}(1-m)(1-\pi)Y[t-1] + \Delta M^S[t-1] \Leftrightarrow \\ Y[t] &= \varphi Y[t-1] + \Delta M^S[t-1] \end{aligned} \tag{6}$$

where we for convenience have introduced a parameter

$$\varphi = 1 - s_{WE}(1-m)(1-\pi) \quad , \quad 0 < \varphi \le 1 \tag{7}$$

Eq. (6) is a recursive equation in $Y[t]$. The recursion is started with an aggregate income $Y[0]$, which is the initial income to capitalists and wage earners due to an assumed initial loan-based investment. We also need a recursive equation for wage earner money hoarding, which is

$$\begin{aligned} M_{WE}[t] &= M_{WE}[t-1] + s_{WE}(1-m)(1-\pi)Y[t-1] \Leftrightarrow \\ M_{WE}[t] &= M_{WE}[t-1] + (1-\varphi)Y[t-1] \end{aligned} \tag{8}$$

We start with no money in the system so that $M_{WE}[0] = 0$, before the initial loan-based investment is made at $t = 0$. $M_{WE}[t]$ for $t = 1, 2, \ldots$, is the entire money stock, since firms and capitalists are assumed not to hoard. Money velocity, or a more appropriate term which will be used in the following, *transaction frequency*, is for wage earners:

$$\begin{aligned} V_{WE}[t] &= \frac{C_{WE}[t] + ms_{WE}Y_{WE}[t]}{M_{WE}[t]} = \frac{(1-s_{WE})Y_{WE}[t] + ms_{WE}Y_{WE}[t]}{M_{WE}[t]} \Leftrightarrow \\ V_{WE}[t] &= \frac{Y[t]}{M_{WE}[t]}(1-\pi)(1-s_{WE}\{1-m\}) \end{aligned} \tag{9}$$

We will now consider two scenarios with money hoarding according to the PK/C model. This means that $m < 1$ in both cases, which again, from (7), implies $\varphi < 1$. The first scenario has no net lending to compensate for hoarding; $\Delta M^S[t] = 0$, $t = 1, 2, \ldots$.

Using (6) repeatedly, we get $Y[t] = \varphi^t Y[0]$ (10)

$Y[t]$ will then $\to$ 0, since $\varphi < 1$. Wage earner money hoarding, using (8) and (10), is

$$M_{WE}[t] = M_{WE}[t-1] + (1-\varphi)\varphi^{t-1}Y[0] \tag{11}$$

Using the recursive property of (11) several times with $t = 1, 2, \ldots$, we get

$$M_{WE}[\infty] = (1-\varphi)\varphi^0 Y[0] + (1-\varphi)\varphi^1 Y[0] + \ldots =$$
$$Y[0](1-\varphi)(1+\varphi+\varphi^2+\ldots) = Y[0](1-\varphi)\left(\frac{1}{1-\varphi}\right) = Y[0] \qquad (12)$$

The last step follows from the summation property of a convergent geometrical series. But we could have found this result in a more intuitive way: The only money in the system is introduced in the form of aggregate income $Y[0]$ in the interval $[0, 1]$. No further money is created. Over time money is persistently siphoned off to the wage earner money hoard. As long as $Y[t] > 0$, money leaks out of the circuit through this irreversible process. The final accumulated hoard must then be $M_{WE}[\infty] = Y[0]$, and all money is taken out of circulation. Transaction frequency, using (10) and (12) in (9) will tend towards zero:

$$V_{WE}[\infty] = \frac{Y[\infty]}{M_{WE}[\infty]}(1-\pi)(1-s_{WE}\{1-m\}) = \frac{0}{Y[0]}(\ldots) = 0 \qquad (13)$$

The system reaches equilibrium, but a completely unrealistic one: all money is hoarded, no money circulates, transaction frequency is zero.

A second scenario is when net lending occurs so that it exactly compensates for hoarding. We set $\Delta M^S[t] = (1-\varphi)Y[t]$. (14)

From (6) we then have: $Y[t] = Y[t-1] = Y[0]$ for all $t > 0$ (15)

Wage earner money hoarding, using (8) and (15), is now

$$M_{WE}[t] = M_{WE}[t-1] + (1-\varphi)Y[0] \qquad (16)$$

$M_{WE}[t]$ increases linearly with *t*; we have growth without limit, $M_{WE}[\infty] = \infty$ (17)

For large *t*, transaction frequency becomes a constant divided by infinity,

$$V_{WE}[\infty] = \frac{Y[0]}{\infty}(1-\pi)(1-s_{WE}\{1-m\}) = 0 \qquad (18)$$

Again the system tends towards an unrealistic end state with zero transaction frequency, but now with an an infinite money hoard.

We will in the next section present a modified model for money hoarding which does not give the above unacceptable results.

3. A better money hoarding model

Consider figure 3, which is a "physical analogy" representation of the PK/C model of the monetary circuit:

----- Figure 3 about here -----

The firm sector is portrayed as a "pipeline", outputting aggregate income Y^i to wage earners ("WE") and capitalists ("K"), who here are lumped together, portrayed as another serially connected pipeline. From this pipeline again, the outflow is split into one flow going to a "vessel", which represents the wage earner money hoard, and the rest Y^{di} – consisting of wage earner and capitalist consumption flows, and investment out of profits – goes back to the input of the firm pipeline. With fresh net investment loans added, it constitutes aggregate demand Y^d, as indicated in the figure. It is obvious that any unit of money introduced to the firm sector as part of Y^d, will need some time before it emerges as income for capitalists and wage earners. This delay is T_F. In the same manner, income will need some time T_{WE} before it emerges as money – is spent – for consumption or investment. (For simplicity, this time delay is assumed equal for wage earners and capitalists.) Note that there is no delay when money is passed *between* sectors (black lines with arrows in the figure) – delays are *within* the two sectors (pipelines). Since the pipelines are serially connected, the total time delay must be

$$T = T_F + T_{WE} \tag{19}$$

But if money needs time to traverse a sector, there must at any time be an amount of money *within* the sector – the "volume in the pipeline". The longer the pipeline, the bigger the volume and longer the time delay, which is why the time delays in figure 3 are indicated along pipeline lengths. *It is these circulating amounts that constitute the hoard(s) of money in the economy – and the "vessel" at the right in figure 3 is superfluous and misleading.* Let us therefore remove it, see figure 4.

----- Figure 4 about here -----

We may say that the model in figure 4 portrays *in-flow hoarding*, while the (wrong) model in figure 3 depicts *branching-off hoarding*. This distinction is a key point of this paper.

We do not for our argument need to distinguish between firms and recipients of income, as indicated. The aggregate circulating money hoard in the economy is now called M (= all money in the "pipeline", whether in the hands of firms, capitalists or households. We therefore dispense with subscript $_{WE}$). The superscript ^{di} in Y^{di} is now justified – it signifies "income-based demand", as opposed to total demand Y^d which also stems from loans. We will now make a multi-period (i.e. time-discrete) model of the system in figure 4 . For money stock we have

$$M[t] = M[t-1] + \Delta M^S[t-1] \quad \Rightarrow \quad M[t] = M[t-1] + Y^d[t-1] - Y^{di}[t-1] \tag{20}$$

Introducing transaction frequency, V, we have[2]

$$M[t] = \frac{1}{V}Y^{di}[t] \tag{21}$$

Eq. (21) substituted in (20) gives $\quad Y^{di}[t] = (1-V)Y^{di}[t-1] + VY^d[t-1]$. (22)

Since, from figure 4, $Y^d[t] = Y^{di}[t] + \Delta M^S[t]$, we get

$$Y^{di}[t] = Y^{di}[t-1] + V\Delta M^S[t-1] \ , \tag{23}$$

which we could have seen directly from the figure: Any $\Delta M^S[t] > 0$ will lead to an increase in $Y^{di}[t]$ since no money leaves the circuit. The amount of increase in $Y^{di}[t]$ in the interval ($t-1$, t) is proportional to V; a large V implies that an injection of fresh money contributes

2. The inverse, $1/V$, has dimension time and is a measure of liquidity preference. It is considered constant here for purposes of simplified presentation, but may very well be treated as a variable, driven by feedback from such factors as output change, the interest rate, mood changes among the public. Then our linear model becomes non-linear. An exploration of this is outside the scope of this paper – the reader is referred to (Andresen, 1999).

strongly to Y^{di}. The other way round, a small *V* leads to a small increase in Y^{di} even if money stock increase is the same. Therefore $1/V$ is a measure of liquidity preference. If $\Delta M^S[t] = 0$, then $M[t] = M =$ constant, and this money stock mediates an output $Y^d[t] = Y^{di}[t] = Y = MV$, that is smaller the stronger liquidity preference is.

3.1 Changing to continuous time

We have until now worked in discrete time. This is a somewhat artificial approach, since transactions occur so frequently that the macro system as a whole should be considered to be continuous. We will therefore derive a continuous dynamic model, corresponding to the equations in the last subsection. We do the transition in the usual way by defining an interval Δt between two adjacent points on the discrete time axis, designated $k-1$ and k (the change of discrete time symbol from *t* to *k* is done because we from now on will use *t* to signify continuous time), and then letting $\Delta t \to 0$. Before doing this, however, we will state some facts about stocks and flows in time discrete versus continuous representation: The entities $Y[k-1], \Delta M^S[k-1]$ have somewhat misleadingly been called “flows” until now. More precisely, they are *amounts* of money, denomination < \$ >, that accrue due to corresponding flows, denomination < \$ / time unit >, *over an implied time period*. (Note: the syntax < > is used throughout to indicate denomination.) The instant *t* in continuous time is now defined to correspond to *k* in discrete time; we have $t = k\Delta t$. We also have $t - \Delta t = (k-1)\Delta t$, i.e. discrete time $k-1$ corresponds to continuous time $t - \Delta t$. We may then, if flows are assumed constant within the interval, write

$$Y^{di}[k] = y^{di}(t)\Delta t\text{, and} \tag{24}$$

$$\Delta M^S[k] = \Delta m^S(t)\Delta t \tag{25}$$

Lower case letters are from now on used to signify that $y^{di}(t)$ and $\Delta m^S(t)$ are flows (as opposed to stocks) in continuous time, with denomination < \$ / time unit >. And ordinary parentheses ( ) mean continuous time, while brackets [ ] signify discrete time.

The money stock in the circuit is $M[k] = M(t)$. (26)

Thus the measure of a stock – denomination < \$ > – is *not* changed when we transit to continuous time, in contrast to a flow.

Now to transaction frequency; using (21) and (24) we have:

$$V = \frac{Y^{di}[k]}{M[k]} = \frac{y^{di}(t)\Delta t}{M(t)} = v\Delta t, \quad \Rightarrow \quad y^{di}(t) = vM(t) \tag{27}$$

V has denomination < \$ / \$ = dimensionless >. We note that while V tends to zero with Δt, v is well-defined for $\Delta t = 0$, with denomination < 1 / time unit >. Now we are ready to derive the continuous model. Using (24) and (27) in (22), we get:

$$y^{di}_{(t)}\Delta t = (1 - v\Delta t)y^{di}_{(t-\Delta t)}\Delta t + v(\Delta t)^2 y^{d}_{(t-\Delta t)} \tag{28}$$

(We temporarily place the time variable in subscript position to avoid confusing these parentheses with other parentheses). Moving $y^{di}_{(t-\Delta t)}\Delta t$ over on the left side, and dividing with $(\Delta t)^2$ on both sides we get

$$\frac{y^{di}_{(t)} - y^{di}_{(t-\Delta t)}}{\Delta t} = v(-y^{di}_{(t-\Delta t)} + y^{d}_{(t-\Delta t)}) \tag{29}$$

Letting $\Delta t \to 0$ we get the linear differential equation

$$\frac{d}{dt}y^{di}(t) = v(-y^{di}(t) + y^{d}(t)) \tag{30}$$

Using the right part of (27) in (30), or the straightforward argument that the derivative of money stock must equal the difference between incoming and outgoing flows, leads to

$$\frac{d}{dt}M(t) = -y^{di}(t) + y^{d}(t) \tag{31}$$

If we close the feedback loop, i.e. use $y^{d}(t) = y^{di}(t) + \Delta m^{S}(t)$, (30) and (31) trivially becomes

$$\frac{d}{dt}y^{di}(t) = v\Delta m^{S}(t) \quad \text{, and} \qquad \frac{d}{dt}M(t) = \Delta m^{S}(t) \tag{32}$$

i.e. when money is injected into the system, money stock is the integral of the injected flow.

We wish to examine the dynamics of the "pipeline" in figure 4. We therefore remove the feedback connection $y^d(t) = y^{di}(t) + \Delta m^S(t)$. Then dynamics is governed by (30). If we inject a unit of money at time $t = 0$, for instance in the form of a loan, this mathematically corresponds to $y^d(t)$ being an *impulse function*, $y^d(t) = \delta(t)$. This function is a mathematical idealisation: It may be defined as the limit of a rectangular-shaped time function

$$\delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t), \text{ with } \quad \delta_\varepsilon(t) = \begin{cases} 1/\varepsilon, & |t| \le \varepsilon/2 \\ 0, & |t| > \varepsilon/2 \end{cases} \tag{33}$$

$\delta(t)$ has infinite amplitude and zero duration, but such that its area is unity. In an economic model in continuous time the impulse function is a useful concept, since it allows a correct representation of discrete events: an amount of money Q received by a sector or an agent at time t_1, is represented as an input function $Q\delta(t - t_1)$. The *impulse response* at the output of a sector given by (30) to one unit of money received at $t = 0$ (represented by the function $\delta(t)$), is found by solving (30) with the input $y^d(t) = \delta(t)$. The result is

$$y^{di}(t) = v e^{-vt} = \frac{1}{T} e^{-\frac{t}{T}}, t > 0 \tag{34}$$

where we have now introduced the *time lag* $T = 1/v$. The output impulse response (34) is shown to the left in figure 5. It is a spending flow with denomination < \$ / time unit >. (Note: the symbol $h(t)$ is reserved in the system-theoretic literature to signify the output response to an *impulse* function, as opposed to responses to other input functions.)

----- Figure 5 about here -----

The time *lag T* is something else[3] than a time *delay T*, which we have used initially to discuss dynamics in the circuit (figures 3 and 4). The model (30), which gives the output

3. The control systems literature distinguishes between these two concepts, while the economics literature often use "lag" for what should strictly be called a *delay*.

impulse response shown to the left in figure 5, will from now on be called the *TL* (= *time-lagged*) *model*. A lag means that an input at any time instant results in an output that is dispersed in time, as indicated to the left in the figure. In contrast, the output impulse response of a pure delay system, indicated to the right, is simply the input impulse itself, delayed by *T* but not dispersed.

The area under both responses is unity. This is because money is not created or destroyed in the “pipeline”. *T* is the mean time lag of the response (34), given by

$$T = \int_0^\infty t h(t) dt \text{ , where } \quad h(t) = \frac{1}{T} e^{-\frac{t}{T}} \tag{35}$$

(The mean lag may be estimated by inspection of the graph for $h(t)$, because *T* is the value of t at the intersection between the tangent of $h(t)$ at $t = 0$, and the time axis.) The TL response also incorporates transaction frequency as $v = 1/T$.

The TL response has an intuitive appeal: if an amount of money is received by a sector at $t = 0$, this amount will be time-dispersed on its way through. Some of it will follow a very convoluted path in the sense that it will be used by many agents for transactions before having passed through. The same holds for money being received by a single agent at a certain time instant: it will not all be spent at once, but spread out over time. The TL model expresses the dispersed character of the response in a reasonable manner, as opposed to the time-delayed response which does not account for dispersion at all. *This dispersion-in-time phenomenon, which holds for all input-output relationships for agents and sectors, invalidates the approach of analysing circuit dynamics by assuming that these unfold in concluded “periods” – which is a common assumption in PK/C analysis.*

Andresen (1998) has shown that the TL model can be given a sound microeconomic foundation. If one models an economy (or a sector) with a large number of individual agents as a network of units transacting with each other in an arbitrarily interconnected manner – which is what goes on in a market economy – the overall output response of the sector to an outside money injection will resemble that of the TL model. This turns out to be the case even when individual agent responses vary widely, both in speed and shape – and even if behaviour changes over time.

As mentioned initially, the TL model is similar to A. W. Phillips' "hydraulic" macroeconomic model. See figure 6, where we show the output response to a stepwise change in input flow. The choice of such a model was in Phillips' times justified like this: The economy *needs time* to adjust to a change in demand, and the first order time lag is the *simplest* model for such dynamics. Thus Phillips and others (for instance Godley and Cripps (1983)) use an "Occam's razor" type of justification for their choice of the TL model.

----- Figure 6 about here -----

Outflow is proportional to money stock through transaction frequency $v = 1/T$, we have $y^{di}(t) = vM(t)$, or $y^{di}(t) = M(t)/T$. (36)

An economy with a high *v* then corresponds to a vessel with a wide aperture at the bottom, so that the level of fluid (the money stock) is low in relation to the size of in- and outflows, and vice versa. An increase in liquidity preference corresponds to constricting the bottom aperture. Fluid level will then rise until pressure is so much increased that outflow again equals inflow, and a new equilibrium is reached. The central parameter is the time lag *T*, or its inverse: the transaction frequency *v*.

4. Simulating a modified model

4.1 Presenting the model

A modified PK/C model (hereafter called "the modified model") will be presented and simulated in the following. We will first give the model as a set of equations, and then relate this set to a block diagram which is equivalent to the equation set (this may be seen as a plea from the author to the economics profession to consider such diagrams as a useful alternative way of portraying dynamic models). The model consists of six interconnected sub-models (sub-systems), each given by a linear differential equation.

In the modified model, erroneous branching-off hoarding is substituted with in-flow hoarding. It is more detailed than the PK/C model. The modified model has a bank sector, to account for debt service due to an initial loan that starts the process. It also allows for an interest rate > 0 and a selected finite loan duration, as opposed to the PK/C model which assumes zero interest rate and that both the initial loan and securities are perpetuities or rolled over.

Firms' investment and consumption out of profits in the modified model are lumped together in a single flow, given by a profit share p of output. This is acceptable since both flows appear as demand at the input of the firms subsystem (sector). Households spend a share c for consumption, the rest is used for purchase of securities.

The bank sector has costs that are expressed through an "expenses coefficient" z, i.e. banks split the incoming debt service flow between re-lending (with share $1 - z$) and paying their expenses (share z). There is a lag before the split so that there is "in-flow" money within the bank sector. Note that – in this exercise – no new money is assumed created after the initial loan: all later loans from banks stem exclusively from the debt service income from earlier loans. The model also has a non-bank financial sector which mediates households' security purchases, with a lag. This sector is for simplicity assumed cost-free (i.e. its "z" is zero) .

We define further variables as follows (all stocks and flows are time dependent, other entities are constant parameters):

M_F, T_F : current money stock in firms sector, and the associated sector time lag.

x_{iF} : input flow to firms sector, earlier called y^d.

x_{oF}: output flow from firms sector (earlier called y^{di}); we have $x_{oF} = M_F/T_F$, cf. (36).

M_H, T_H: current money hoard held by households, and the associated sector's time lag.

x_{iH} : input flow to households (wage earners) sector.

x_{oH}: output flow from households, $x_{oH} = M_H/T_H$.

M_B, T_B: current money stock in bank sector, and associated sector time lag.

x_{iB} : input flow to bank sector.

x_{oB}: output flow from bank sector, $x_{oB} = M_B/T_B$.

M_S, T_S: current money amount (stock) in transit for purchase of securities, and the associated non-bank financial sector time lag.

x_{iS} : input flow to non-bank financial sector; for purchase of securities.

x_{oS}: output flow from non-bank financial sector = households' loans to firms, $x_{oS} = M_S/T_S$

D_B, T : firms' current debt to banks, and duration of loans (which for simplicity is assumed to be the same for all loans).

x_{iD_B}: loan flow from banks = input flow to banks' *debt-service generating subsystem*. This will be explained further below.

x_{oD_B}: output from banks' debt service generating subsystem = bank debt service flow imposed on firms.

D_S, T: firms' current debt to households, and time to maturity of associated securities (for simplicity assumed to be the same as bank loans' duration).

x_{iD_S}: input flow to the securities-related debt service generating subsystem = households' loans to firms = x_{oS}. Explained below.

x_{oD_S}: output from securities-related debt service generating subsystem = debt service flow imposed on firms due to households holding securities.

a: we assume an annuity repayment scheme for loan and securities repayment, such that debt service flows are aD_B and aD_S, respectively. More about a below.

$Q\delta(t)$: The initial bank loan to start the system at $t = 0$ is Q; in continuous time represented by an impulse function.

Initial values for all M's and D's are assumed to be zero, i.e. there is no money or debt in the system before the initial loan Q is injected.

Using the defined entities above, the equations for the total system are:

$$\dot{M}_F = x_{iF} - x_{oF}\text{, with } x_{oF} = M_F/T_F$$
$$\text{and } x_{iF} = x_{oB} + x_{oS} + cx_{oH} + p(x_{oF} - x_{oD_B} - x_{oD_S}) \tag{37}$$

$$\dot{M}_H = x_{iH} - x_{oH}\text{, with } x_{oH} = M_H/T_H$$
$$\text{and } x_{iH} = x_{oD_S} + (1-p)(x_{oF} - x_{oD_B} - x_{oD_S}) = px_{oD_S} + (1-p)(x_{oF} - x_{oD_B}) \tag{38}$$

$$\dot{M}_B = x_{iB} - x_{oB}\text{, with } x_{oB} = M_B/T_B \text{ and } x_{iB} = x_{oD_B} \tag{39}$$

$$\dot{M}_S = x_{iS} - x_{oS}\text{, with } x_{oS} = M_S/T_S \text{ and } x_{iS} = (1-c)x_{oH} \tag{40}$$

$$\dot{D}_B = x_{iD_B}\{t\} - x_{iD_B}\{t-T\}\text{, with } x_{oD_B} = aD_B \text{ and } x_{iD_B} = (1-z)x_{oB} + Q\delta(t) \tag{41}$$

$$\dot{D}_S = x_{iD_S}\{t\} - x_{iD_S}\{t-T\}\text{, with } x_{oD_S} = aD_S \text{ and } x_{iD_S} = x_{oS} \tag{42}$$

Time dependency is not shown except in the last two equations, where it is indicated with braces { }. (41) and (42) will be explained below. All the *x*'s may be eliminated from (37) – (42), and the resulting set is

$$\dot{M}_F = -(1-p)M_F/T_F + M_B/T_B + M_S/T_S + cM_H/T_H - p(aD_B + aD_S) \tag{43}$$

$$\dot{M}_H = -M_H/T_H + paD_S + (1-p)(M_F/T_F - aD_B) \tag{44}$$

$$\dot{M}_B = -M_B/T_B + aD_B \tag{45}$$

$$M_S = -M_S/T_S + (1-c)M_H/T_H \tag{46}$$

$$\dot{D}_B = \frac{1-z}{T_B}(M_B\{t\} - M_B\{t-T\}) + Q(\delta\{t\} - \delta\{t-T\}) \tag{47}$$

$$\dot{D}_S = \frac{1}{T_S}(M_S\{t\} - M_S\{t-T\}) \tag{48}$$

This is a 6. order coupled system of linear differential equations. The last two equations contain time-delayed variables, which means that a closed algebraic solution for the system cannot be found. But as we shall see, the system may be numerically simulated.

Equations (43) – (46) have TL dynamics. This has already been discussed. It remains to explain equations (47) and (48), or equivalently: (41) and (42). These are the equations for “the debt service generating subsystems”. Both subsystems are assumed to have precisely the same dynamics, the differences are only due to different input flows. We will examine these subsystems’ impulse response. With an impulse $\delta(t)$ as input, (41) or (42) turns into

$$\dot{D} = \delta(t) - \delta(t-T) \tag{49}$$

We here use a generic symbol “*D*” for debt. D is then the integral of the two impulses on the RHS of (49). Since an impulse per definition has area = 1, an integral $\int_{-\infty}^{t} \delta(\alpha - T)d\alpha$ is

$$\int_{-\infty}^{t} \delta(\alpha - T)d\alpha = \begin{cases} 1, & t > T \\ 0, & t < T \end{cases}\text{, which is a \textit{unit step function}.} \tag{50}$$

Integrating each RHS term in (49) gives unit step functions as indicated with dotted lines in figure 7:

----- *Figure 7 about here* -----

The integrated first term gives a positive unit step function that jumps to 1 at $t = 0$, the second function jumps to -1 at $t = T$. D is their sum, as indicated with the thick shaded line. Multiplying by a factor *a* gives a constant debt service flow aD that is cut off at the end of the loan’s duration. This corresponds to an annuity repayment scheme. It now remains to derive *a*. The usual way to calculate an annuity payment is to do it for discrete time, for payments that are due for instance each month. Since we work in continuous time, we assume a continuous constant interest + principal *flow* aD, equivalent to the discrete time scheme. The interest rate is *i*. We demand that *a* is such that the present value of the flow aD between 0 and *T* is equal to D:

$$\int_{0}^{T} aDe^{-it}dt = D\text{. From this follows } a = \frac{i}{1 - e^{-iT}} \tag{51}$$

(We easily ascertain that for the special case when the loan is a perpetuity, i.e. $T = \infty$, eq. (51) gives $a = i$ as expected. For the special case $i = 0$, L’Hopital’s rule gives $a = 1/T$, also as expected.)

By this our model is complete. The chosen annuity scheme could be interchanged with a bond-type repayment scheme, where principal is only paid at the end of the loan's duration. Extensive simulations have shown that such a change does not impact significantly on results, so we will stick with the annuity scheme.

The modified model may equivalently be portrayed through a block diagram. See figure 8.

----- *Figure 8 about here* -----

The block diagram is from the simulation software package *Simulink* (The MathWorks Inc., 2000). The advantages of block diagrams are twofold: first, it is easier to see the interactions between the different parts of the system through such a diagram, than through a set of equations. Second, today's block diagram-based tools make it very fast to implement a model: by selecting, dragging, dropping and then connecting different elements using the mouse, one does the equivalent of setting up differential equations for numerical simulation.

The diagram in figure 8 is interpreted like this:

- Big rectangles are *subsystems* (more below).
- Narrow upright rectangles with plus or minus signs signify the same as the summation circles introduced earlier: A variable leaving such a rectangle is the sum of variables entering the same rectangle.
- A triangle signifies multiplication of an incoming variable with the constant in the triangle.
- Rectangles with incoming arrows but no outputs, signify storage of variables from the simulation. The time axis is defined by a "clock" which outputs time with (chosen here) monthly intervals. All other variables are also stored with monthly intervals.
- Rectangles with no incoming arrows, only output arrows, are either constants or

exogenous time-varying functions that influence the system.

The “firms”, “banks”, “lag for household security purchases” and “households” subsystems are all TL models as already discussed, and block diagrams are therefore not shown. The last two blocks are the debt service generating subsystems already discussed.

By now we are ready to present different simulations.

4.2 Simulation with the modified model

As a first step we want to check whether it is true that the system cannot sustain without net money being created through bank loans to firms, to supplement investment out of retained earnings and sales of securities to households. This is the claim of Fontana et al., and follows from their assumption of wage earners doing “branching-off” hoarding. But it turns out that the more correct model[4] with in-flow hoarding manages to reach stable equilibrium without new money injections, under wide ranges of parameter values. Figure 9 shows a representative simulation run. This defines a “benchmark case” which will be perturbed in further simulations.

----- Figure 9 about here -----

We first consider the upper part of the figure. Note the time span, which is 20 years = 240 months. The parameter values chosen are shown in the upper left-hand corner. The interest rate *i* is set to 5% per year which is high for a system with no new money creation. Profit share of output is set to 0.2 . The propensity to consume *c* for wage earners and the “expenses share coefficient” *z* for banks are both set to 0.7 . The time lags for households

4. The reader is warned that both this modified version, and the original PK/C model with branching-off hoarding, are both somewhat unrealistic in the sense that a homogenous group of households – wage earners – are assumed to be the main source of savings, and there are no rentiers. This model is used all the same, since the main point of this paper is to examine the effect of replacing branching-off hoarding with in-flow hoarding. If one does the same exercise with non-saving workers and rentiers who save, one gets similar results.

(wage earners), firms, banks and securites sales are respectively set to 1, 3, 2 and 2 months. The choices of these time lag values are not critical – dynamics are essentially the same as long as time lags in sectors are small (in the order of months) in relation to the much slower dynamics of the debt service subsystems (in the order of 5 years and more).

We observe that after a period of slightly more than 5 years – the chosen loan duration time – the system settles down in an equilibrium state. What happens is the following: The initial loan received by firms is – with lag T_F – paid out as profits or wages. In a very short period, a couple of months, corresponding to the time lag of the wage earners' subsystem, the necessary hoarding level there is reached and the outflow from wage earners has risen from zero to a value nearly equal to that of the incoming wages flow. What happens in this first fraction of a year is shown in the lower part of figure 9, where the time axis near $t = 0$ is strongly magnified (the same ellipse is shown in both graphs to indicate time scales). Part of the outflow from wage earners is consumed, another part passes through the securities subsystem which again outputs loan flows. Loan flows also emerge from banks based on their receiving income from debt service on the outstanding initial, and successive, loan(s). But a share *z* from banks is used to pay for their expenses. The flows from from securities sales, and from banks, become nearly equal to the two financial subsystems' incoming flows, after a short (months) build-up of a hoard there – corresponding to the time lags of these subsystems. The loan flows cause two corresponding debt service flows that are deducted from output before the rest is shared between firm owners and households. Debt service starts immediately after a loan is paid out, and since we have a continuous flow of new loans, this explains the positive slope of the debt (service) flow for $t < 5$ years. But for $t = 5$ years the debt burden from the large initial bank loan is terminated. The effect may be observed as a dip in the graphs. And for $t > 5$, debt burdens from successively maturing bank loans and securities are removed as time goes. This nearly cancels out the effect of new debt service commencing because of new loans. After a short time the cancellation is near perfect. The system has transited to a phase where all flows are in equilibrium. Now new loans are paid back at a rate that is sufficient to avoid further accumulation of debt and corresponding increase in debt service for firms. And money hoards are constant for households, firms and banks.

These results contradict the following: “... the liquidity owned by wage earners is in the nature of a residue ... At most, through the issue of financial assets, firms are able to replenish part or all of the existing liquidity and thereby to replenish bank balances partly or completely. At any given moment, there is only finance to repay banks but not fund new businesses.” (Fontana 2000, p. 37). But the liquidity held by any sector is not a *residue*, but remains as a reservoir “*in the flow*”. These reservoirs reach some maximum level and do not build up further. Then part of the constant circulatory flow may be used to extend new loans both to current and new businesses, ad infinitum.

Lavoie writes: “The debt of firms *vis-à-vis* banks must increase from period to period, unless households decide to diminish their bank deposits.” (Lavoie, 1992, 156.) This is also contradicted by the above results. The system reaches an equilibrium where households have a constant money hoard, and firms have a constant debt. New loans are extended, but the effect of this is neutralised by old loans being repaid.

Graziani writes: “To the extent that wage earners, instead of spending their whole incomes either on commodities or securities, keep their savings in bank deposits, the firms are unable to repay their banks debts. If the previous level of output has to be preserved, the firms have to get new loans from the banks, which means that the ‘stock’ of money has increased.” (Graziani, 1996, 144). He is correct that loans must be given to compensate for existing ones being re-paid, if circulation is to continue. But as long as this happens, output may be upheld with zero net lending (i.e. no increase in money stock) if wage earners do in-flow hoarding, as shown in the simulation. Graziani’s remark about the neccessity of increasing money stock is logical only in the framework of the erroneous branching-off hoarding model.

Another claim based on the PK/C model is that "every repayment of credit.....must immediately be re-lent if activity is to be maintained" (Davidson, 1987, 151, quoted in Lavoie, 1992, 157). This turns out not to be the case for the model with in-flow hoarding. Consider the simulation results shown in figure 10:

----- *Figure 10 about here* -----

Parameters are identical to those in figure 9, except that the two financial sectors' time lags are set five times larger, $T_B, T_S = 10$ months, to check what happens when Davidson's "immediately" is far from being satisfied. If we compare figures 9 and 10, we observe that the dynamics are similar, and both systems end up in equilibrium somewhere above 10 years. The difference is that money flows and stocks for firms and households are lower. This is because a larger amount of money is at any time bound up in the financial sectors. But activity is maintained on a constant level after some initial period, just as in the prevous simulation run with $T_B, T_S = 2$.

Figure 11 shows four perturbations of the benchmark case. The parameters that are different from the benchmark set are shown on a black background for each case.

----- *Figure 11 about here* -----

If we compare with figure 9, figure 11 a) shows that a longer loan duration T means somewhat heavier debt burdens in equilibrium. Figure 11 b) shows that – as expected – a higher interest rate implies heavier debt burdens. What may seem surprising, however, is that the system is still sustainable at an interest rate as high as 20%. Figure 11 c) shows that a lower propensity to consume (and a lower "leakage" for expenses from banks, i.e. banks' income flow is to a higher degree re-lent) has a stronger impact towards heavier debt burdens than an increased interest rate. Figure 11 d) is a combination of the perturbations from b) and c): now we have both an unrealistically high interest rate, and an unrealistically small propensity to consume and bank expenses' share factor. The loan duration is also longer than for the benchmark case. Now the system collapses, with debt service surpassing output at around $t = 7$ years. Note however, that the system is still stable in the mathematical sense, values approach equilibrium even if the solution is unacceptable in the real world. (Even stronger parameter perturbations in the same direction – results not shown here – make the system unstable also mathematically, and we get an exponential blow-out.)

By now we have seen that even with high non-zero interest rates, with long duration of loans, and with low values of the propensity to consume / banks' expenses share – the system may still reach stable equilibrium.

5. Conclusions

- In-flow hoarding should be substituted for branching-off hoarding in monetary circulation models. Branching-off hoarding is not only a poor approximation of the real world – any model is after all an approximation – it is fundamentally erroneous.
- The monetary circuit cannot *solely* be analysed verbally. To achieve a correct understanding of the time path of relevant variables one must (also) use differential (or difference) equations. This also enables accounting for the effect of interest on loans, which is usually abstracted from in verbal circuit analysis.
- The first-order time lag is the simplest way to model in-flow hoarding. It gives a reasonable representation with only one parameter, and with the bonus that this parameter – the time lag – is the inverse of transaction frequency for the agent or sector in question. The total monetary circulation system may then be considered an interconnected network of such time lag units.
- Describing the monetary circulation system's dynamics by assuming that an amount of

money that is input at some instant may be accounted for at the end of some “period” is an unacceptable approach, because of the dispersion-in-time effect whenever money passes through a sector or by an(y) agent.

- When discussing conditions for debt-related sustainabilty of a macroecomic circuit, it is not enough to consider the effects of the interest rate, the propensity to consume, the profit share of output etc.: One should also account for banks “leaking” out income from debt service to pay for expenses, as opposed to re-lending all of this income. This is the coefficient z introduced here. A large z plays a similar stabilising role as a large propensity to consume.
- The system-theoretic toolbox, originally developed in the control systems and signal processing environment, is very suitable for modeling and simulation of monetary circulation systems.

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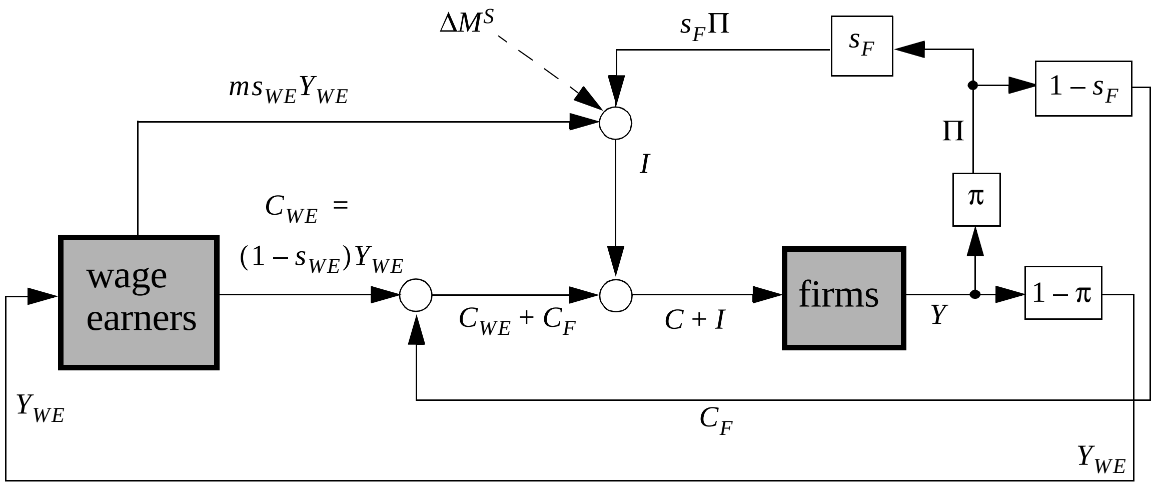

Figure 1: PK/C model with wage earners and firms

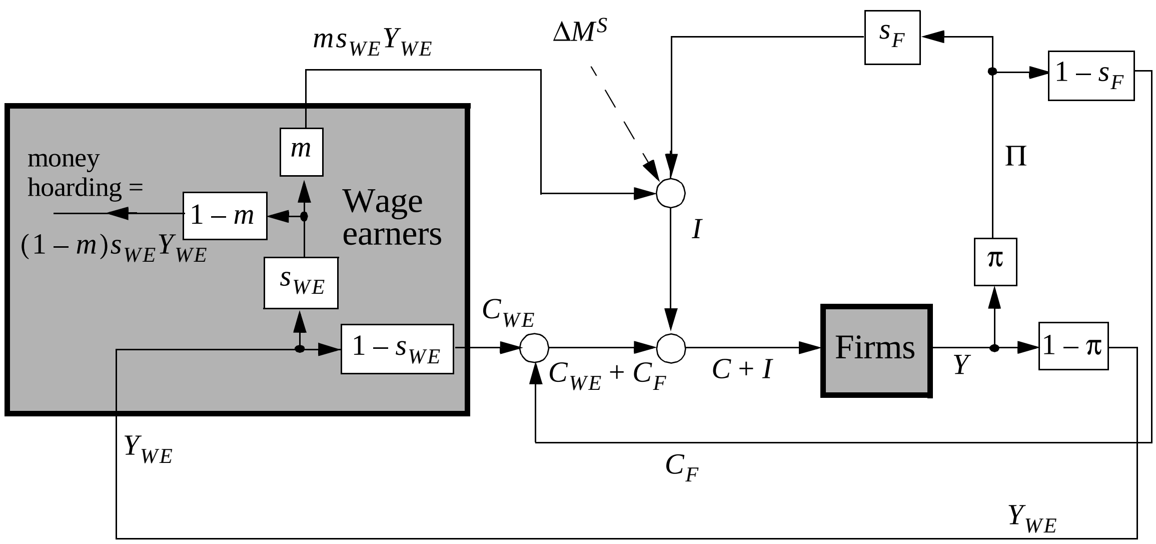

Figure 2: PK/C model with assumed hoarding mechanism

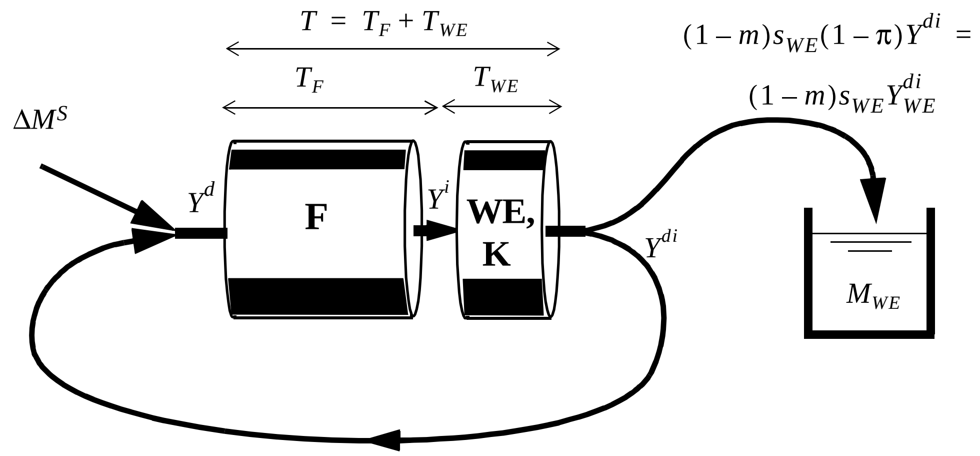

Figure 3: System with branching-off hoarding

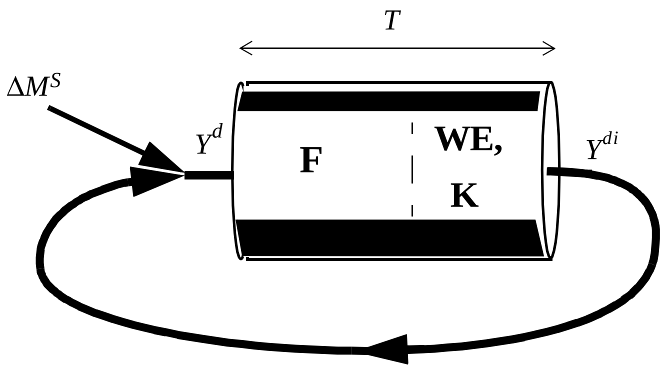

Figure 4: System with in-flow hoarding

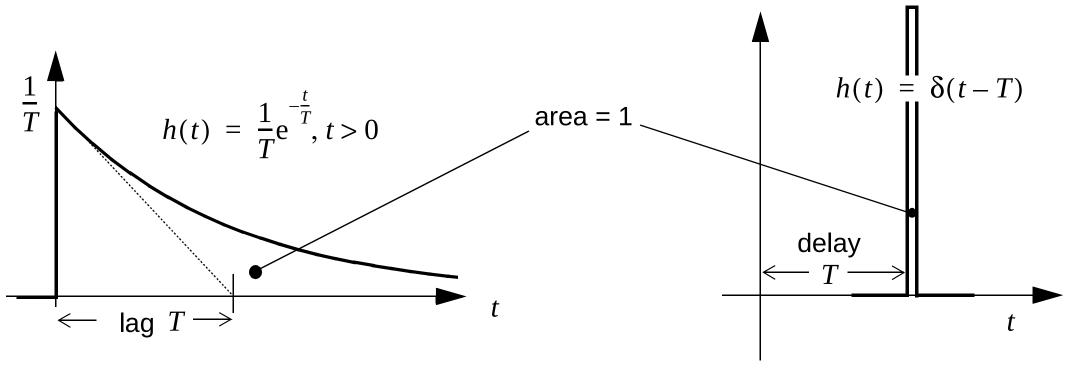

Figure 5: Output impulse responses with time lag (left) and time delay (right)

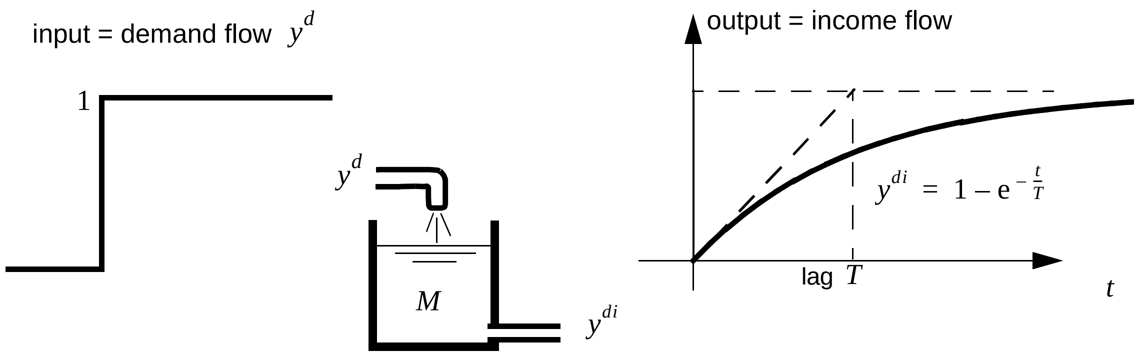

Figure 6: A vessel analogy

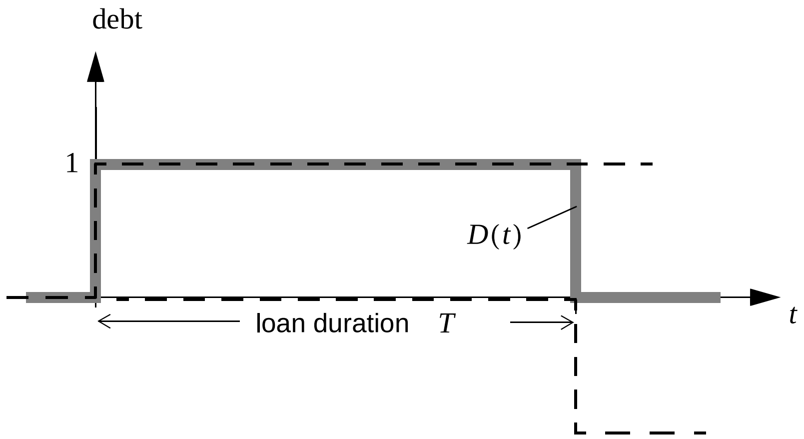

Figure 7: The impulse response of a debt service subsystem is $aD(t)$

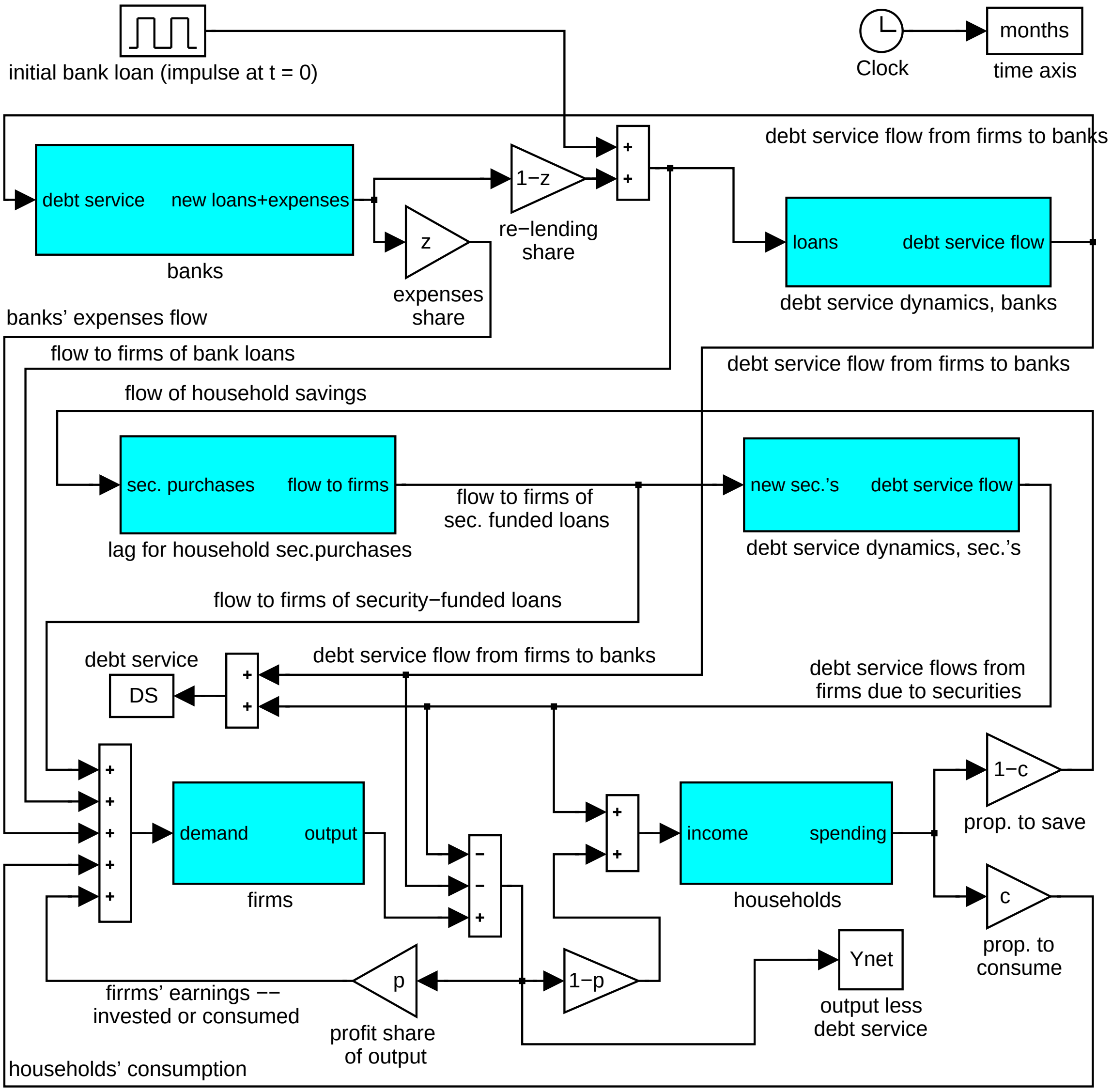

Figure 8: Simulink block diagram of the modified model

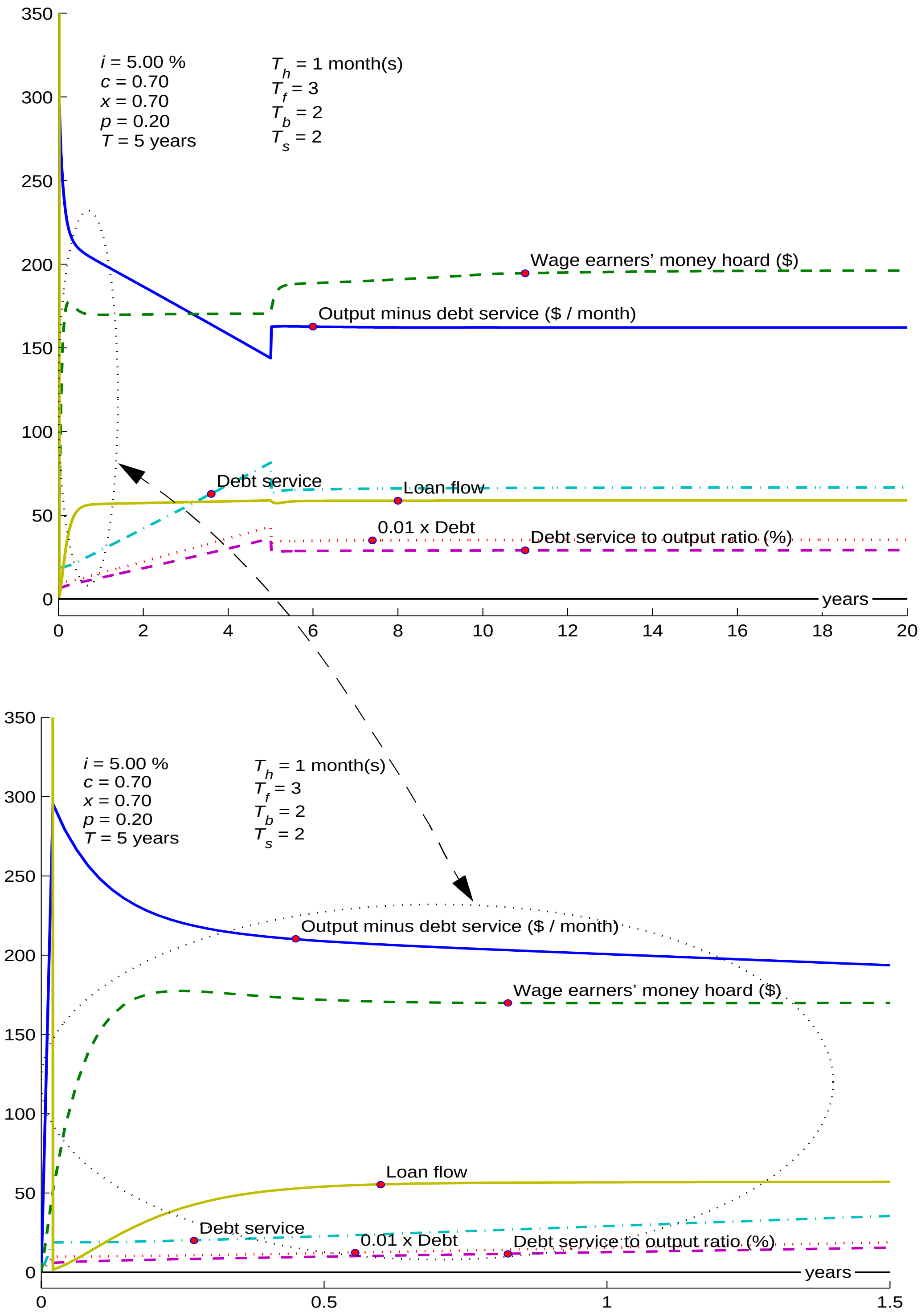

Figure 9: Time paths of an economy with in-flow hoarding

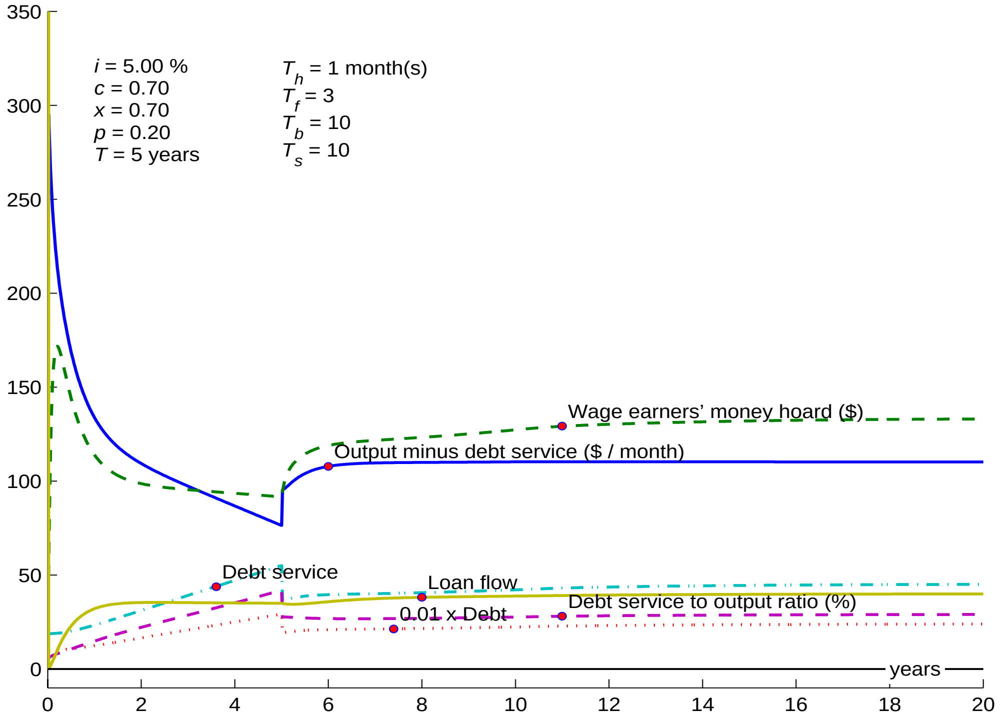

Figure 10: Time path with large lags in financial sectors

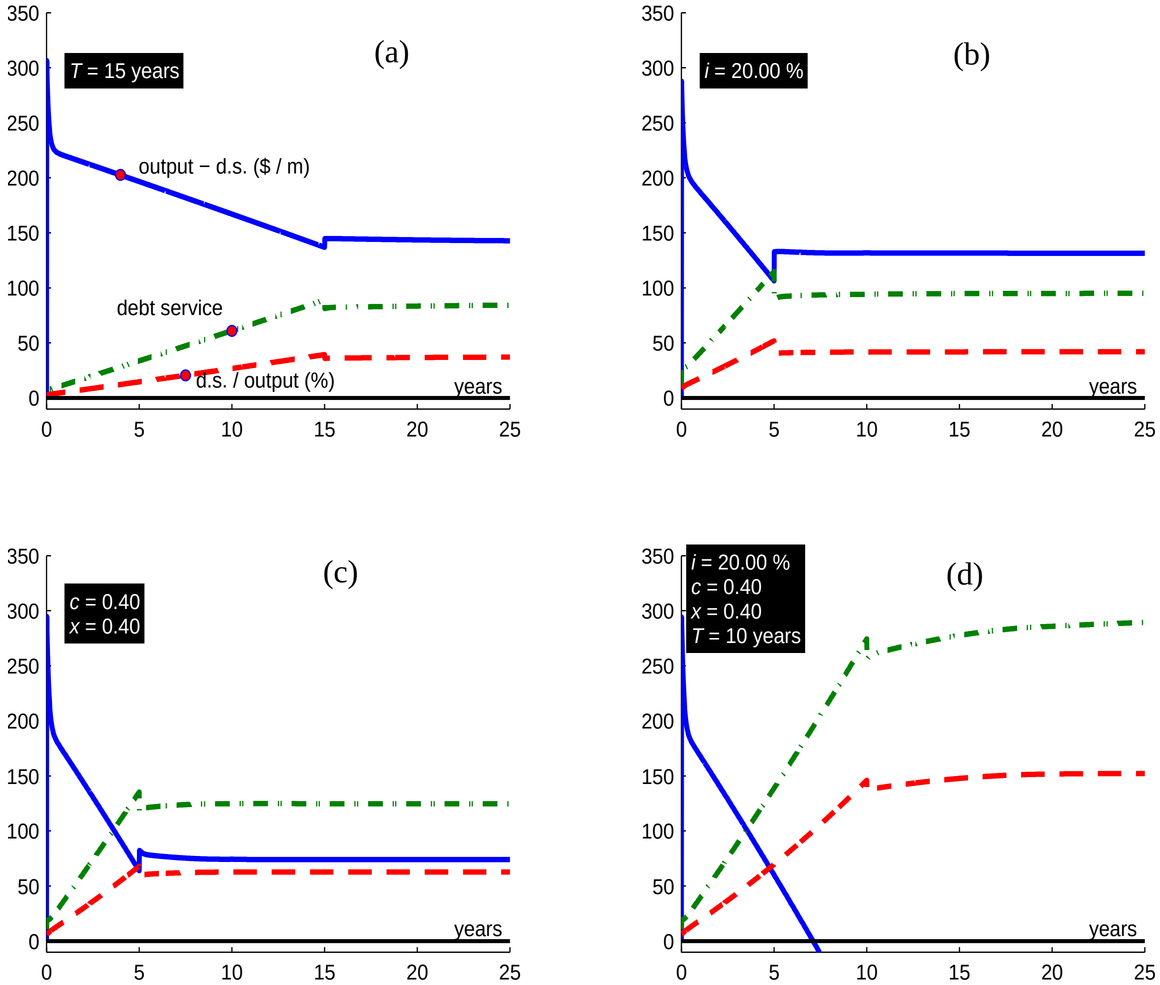

Figure 11: Four perturbations of the benchmark case