Fundamental financial accumulation dynamics

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Abstract

Any economic system with interest on money lent has the potential to gradually develop a level of debt that leads to crisis. Parameters and simple models for the dynamics of financial accumulation are proposed and explored. It turns out that concepts from linear control systems theory, and continuous-time representation, is quite useful for this exercise. It is argued that the problem of "exploding" debt is grave and largely ignored.

Keywords: accumulation, instability, lending, compound interest, dynamics

JEL classification: B50, C02, C60, C67

1 Introduction

This paper discusses the basic mathematical conditions for financial accumulation. The model consists of a "moneylender" who re-lends part of financial inflows from debt service on existing loans so that future financial income will be larger. At the other end is an agent who is in debt but still borrows what the moneylender offers. The two units may be thought of as macroeconomic aggregates, so that we have a society which is polarised between a group of lenders and a group of borrowers (called "sectors" in the following). The term "bank" will be used in between for the moneylender, but this is not a bank in the modern interpretation of the term (regulated by the Basel accords and thus able to create net credit money (Andresen, 2008)) but an entity that only re-lends received money that is left over after the lender has paid his expenses including wages. In this sense the lender corresponds to the "moneylender" of antiquity, among other places criticised in the Old Testament. Since ancient times there has been awareness of the instability inherent in a system where agents re-lend part of their income from loans. This is the rationale for periodic debt forgiveness ("jubilee") as proscribed in the Bible. The reader is referred to Appendix B for some quotations, and to (Hudson, 2009).

This paper discusses these dynamics using concepts from control systems theory. Time is continuous, and money flows are assumed to be smoothly varying in time, even if the actual money "flows" between agents occur as time-discrete events. This assumption is considered acceptable on the time scale (years and decades) we are considering. More on this below, and in Appendix A.

2 The model

A sector receives a money flow contributing to the sector's stock of money. The inflow and the current stock of money basically decides the sector's outgoing flow to other parts of the economy – its spending. But there are also outflows that are not decided by the sector (or agent) in question, but imposed on it from other parts of the economy. Such flows will be termed *non-discretionary*. Taxes or debt service are examples of non-discretionary flows.Taxes are dynamically unimportant since that type of payment occuring at some moment implies no future related flows. Debt service flows, however, have interesting dynamics that unfold over time: an initial one-shot input (received loan) leads to a stream of future events (debt service outflows).

A loan may in continuous time be considered an impulse (a delta function) input to a unit, and then the opposite-sign debt service flow becomes the impulse reponse (more on this in Appendix A) of what we will term a *debt service subsystem (*from now on abbreviated "DSS"*)*. This impulse response is a non-discretionary flow. The model may be explained via the block diagram in figure 1. In the lower part

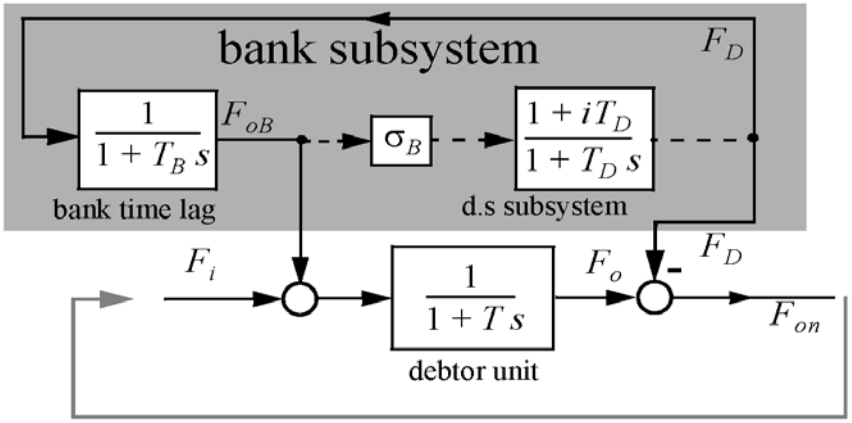

Figure 1: A bank subsystem with recycling of loans

of the figure is a debtor unit[1], which may be a single agent or a sector. If the unit is the entire aggregate of firms and households, F_{on} (= net outflow after debt service F_D) is recycled to the input as the flow F_i, indicated by the shaded arrow lowermost in the figure. In this case the outflow F_o corresponds to the country's GDP.

[1] The variable s in the diagram is a differentiation operator, so that that the lower block $\frac{1}{1+Ts}$ corresponds to the linear differential equation $T\frac{d}{dt}F_o(t) + F_o(t) = F_i(t) + F_{oB}(t)$ $<=>$ $\frac{d}{dt}F_o(t) = -\frac{1}{T}F_o(t) + \frac{1}{T}\left[F_i(t) + F_{oB}(t)\right]$.

F_D is different from other flows in the figure in the sense that it is the result of a *rule* (the loan contract). This rule imposes – it is non-discretionary – a flow F_D on the indebted sector, which is subtracted from the gross debtor outflow F_o, and inserted as an inflow to the bank unit (the aggregate of all lenders). Note direction of arrows. By this the accounting remains correct: money removed from one flow is input somewhere else. To indicate the presence of rule-based interaction as opposed to a sector's own-decided outflow, the corresponding lines are dotted in the figure.

We have here assumed a scenario where the flow of new loans is a strict feedback from what banks receive in debt service on current loans. This is pure lender-controlled financial accumulation. (New loans may instead be mood-dependent and not directly decided by what inflows banks receive, but this is not considered in this paper.)

We will use the term "'bank" here in a quite generic sense: any type of unit that has any type of financial claim (here called a "loan") on another unit/sector as long as the claim obliges the debtor to furnish a future stream of returns. The interest rate is i and duration of loans is T_D . As mentioned earlier, debt service is modeled as a continuous flow, while in the real world debt service occurs as time discrete packets. In our continuous-time setting this could have been precisely accounted for by a train of delta functions, but this is not necessary, following the above argument about the long time scale for the dynamics to be discussed, and also the low-pass filter property of the sectors in the system.

The model in figure 1 has a great advantage: It allows for calculating the dynamics of an aggregate economy where current debt service is used continuously to extend new loans, and where both the effect of interest rate and loan duration is accounted for. This is in contrast to much of literature of the Post Keynesian and Circuitist economic schools, where one often – due to the inferior tools used – has to abstract from interest and also assume that loan extension and repayment takes place in distinct and concluded "rounds", see for instance (Lavoie, 1992) pp. 151 – 157, (Graziani, 1996), and (Fontana, 2000). This topic is treated more extensively in Appendix A.

The (aggregate) "bank" in the figure is modeled as a first order linear system with unity gain, assuming that the flow received by the bank is output again with some lag T_B.These first-order linear dynamics implies that the money held by the bank[2] is $M_B = F_{oB}T_B$. Thus M_B is the state variable of the bank subsystem. The outflow F_{oB} consists of both the bank's paying for expenses, and its new loans flow which is its financially reinvested share σ_B of F_{oB}, $0 < \sigma_B < 1$. We will from now on call σ_B the *financial reinvestment coefficient*, abbreviated *FRC*. The real economy (debtor unit) is – like the bank – modeled as a first order linear dynamic system with unity gain.

It now remains to explain the DSS in the figure. The transfer function is

$$G_D(s) = \frac{1 + iT_D}{1 + T_D s} \tag{1}$$

which may be discussed by introducing the equivalent structure shown to the right in figure 2. Now debt

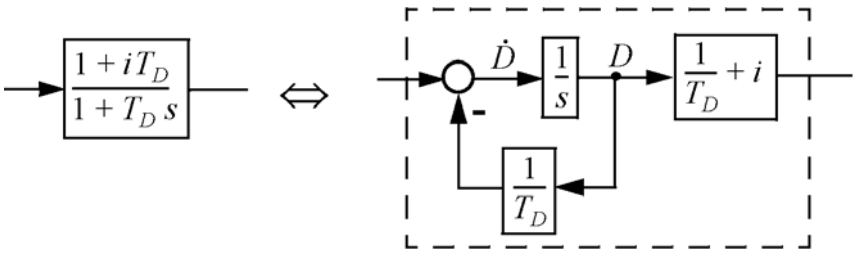

Figure 2: Equivalent debt service block

D is visible in the right subsystem. This DSS, with inflows and outflows as in figure 1, corresponds to the equations

$$\dot{D} = \sigma_B F_{oB} - \frac{1}{T_D} D \tag{2a}$$

$$F_D = (i + \frac{1}{T_D}) D \tag{2b}$$

[2]Some readers may object that the concept of banks "holding money" is meaningless, since banks may be considered to create money when lending, and destroy money when loans are repaid. This is the Post Keynesian position, which this author supports. But it is for purposes of simplified presentation convenient to assume that the bank works like a non-bank financial institution ("moneylender"), in the sense that it does not net create money. This also allows us to include other types of accumulating units in our "extended bank concept".

This scheme (from now on called the "exponential debt service" scheme) is unconventional, since both the principal and interest flow components are proportional to remaining debt. This differs from for instance an annuity scheme where the sum of principal and interest is constant, or a bond-type scheme where principal is only paid (in its entirety) when the loan matures. The advantage of the scheme (2) is that it allows for analysis using eigenvalues, and finding algebraic solutions – while annuity or bond-type dynamics involve time delays and are therefore algebraically less tractable. And it will be demonstrated in subsection 2.3 that differences in total system behaviour are unimportant in regard to which scheme is assumed. Figure 3 shows the debt service flows for the exponential debt service scheme compared to the

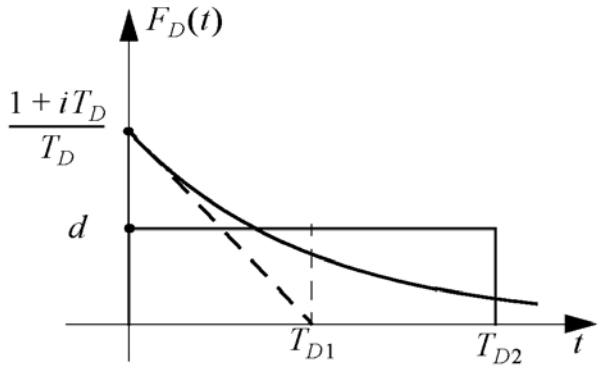

Figure 3: Debt service for annuity and exponential schemes

annuity-type scheme. If we consider a loan of 1 \$ extended at $t = 0$, these debt service flows will be the impulse responses of the debt service subsystems. For approximate equivalence, we suggest that both types of DSS should have the same mean lag. This means that loan durations differ, with $T_{D2} = 2T_{D1}$ (this multiplicative factor will be somewhat adjusted in subsection 2.3). Mathematically, the duration of the exponential debt service scheme is infinite, but we define it to be T_{D1}, since this is the mean lag of the graph. The areas under the graphs correspond to the accumulated debt service sums. They are > 1, so the DSS does not have an impulse response with unit area (it would have had that if i was 0, since then one had to pay back only what was initially borrowed). The value of the constant parameter d in the figure, which gives the annuity debt service flow, is derived below.

2.1 The annuity-type debt service subsystem

We assume that a loan of 1 \$ is extended at $t = 0$, and demand that the discounted value of a received constant flow d between 0 and T_D shall be equal to 1:

$$d \int_0^{T_D} e^{-it} dt = 1, \text{ which gives } d = \frac{i}{1 - e^{-iT_D}} \tag{3}$$

If the loan is a perpetuity i.e. $T_D = \infty$. (3) then gives $d = i$ as expected. For the special case $i = 0$, L'Hopital's rule, or the integral in (3), gives $d = 1/T_D$, also as expected.

We may now construct a subsystem for this annuity type DSS that has a rectangular impulse response with amplitude d. It is shown in figure 4. The subsystem works like this: A new loan (an impulse) is

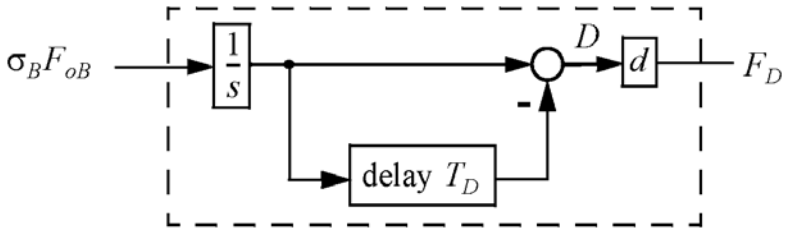

Figure 4: Debt service subsystem with annuity scheme

received, and the integrator makes the value in the upper branch jump to the size of the loan and stay there. After a duration of T_D, the lower branch jumps to the same level and is subtracted from the upper

branch value. This ensures that debt service for that particular loan stops when the loan terminates. We have a rectangular response with amplitude equal to the size of the loan, multiplied with the factor d to give the correct debt service outflow from the DSS.

This DSS contains a time delay, and closed algebraic solutions of systems containing time delays is generally not possible. But the system is still linear. Therefore a continuous flow of new loans will, by convolution with the DSS impulse response, still give the precise debt service outflow. In other words: the effect of continuous recirculation of loans in a macroeconomic model may be correctly accounted for also in the annuity case. And we will see below that in this special case stability may be checked algebraically in spite of eigenvalues not being available.

2.2 When may debt "explode"?

A widely covered topic in literature and a persistent political-economic, moral and religious issue since ancient times is the mechanism of lenders accumulating financial claims on the rest of society by re-lending income from current loans. This danger is recognised for instance in the Bible, where a "jubilee" is proscribed every 50ieth year to reset outstanding debt to zero (see Appendix B).

Obviously, a persistent re-lending of debt servoce flows *may* lead to financial debt/asset polarisation in a society. The structure in figure 1 allows us to check the conditions for this occuring. Debt/asset polarisation corresponds to instability of this linear system. If we initially confine ourselves to a system with an exponential debt service scheme, stability may be checked by considering system eigenvalues. By inspection of figure 1, we see that system dynamics is decided entirely by the shaded "bank" part of the structure. The dynamics of the lower "debtor" part does not feed back to the bank part and is therefore decided solely by what happens there. The characteristic equation for the bank part is

$$(1 + T_B\lambda)(1 + T_D\lambda) - \sigma_B(1 + iT_D) = a_2\lambda^2 + a_1\lambda + a_0 = 0 \tag{4}$$

A necessary (and for a second order system like this, also sufficient) condition for the system's eigenvalues to be negative (i.e. stable system) is that all coefficients a_k in the characteristic polynomial have the same sign. a_1 and a_2 are always positive, while $a_0 = 1- \sigma_B(1 + iT_D)$ may be < 0 for certain parameter values. Then one eigenvalue is in the right half plane. We have instability (= debt growth = financial accumulation). The condition $a_0 < 0$ corresponds to:

$$\sigma_B > \frac{1}{(1 + iT_D)}, \text{ or equivalently:} \tag{5a}$$

$$i\sigma_B > \frac{1 - \sigma_B}{T_D}, \text{ or} \tag{5b}$$

$$iT_D > \frac{1 - \sigma_B}{\sigma_B} \tag{5c}$$

We note that T_B is not part of the instability condition. If the condition (5) is fulfilled, debt growth is exponential (after an initial transient period due to the other, stable eigenvalue). Loan duration T_D may be in the order of – say – a decade. The bank time lag T_B should realistically be in the weeks/months range. So we may assume $T_B \ll T_D$. This means that the bank time lag subsystem in figure 1 may reasonably be substituted by unity. If we also ignore the debtor subsystem which has no impact on dynamic properties as already mentioned, the simplified remaining system needed to discuss debt build-up dynamics becomes as shown in the block diagram to the left in figure 5. To the right we

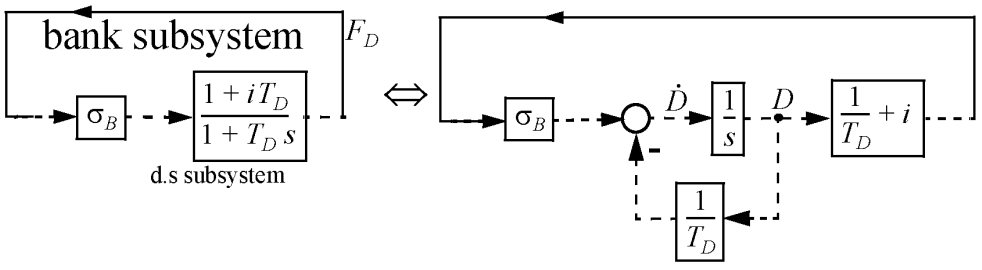

Figure 5: Simplified accumulation system

have inserted the equivalent DSS from figure 2 so that the sole system state, D, is shown. This block

diagram corresponds to the autonomous first order linear differential equation

$$\dot{D} = \left(-\frac{1}{T_D} + \sigma_B (\frac{1}{T_D} + i) \right) D = \lambda D \tag{6}$$

which has the solution $D = D_0 e^{\lambda t}$, where D_0 is initial debt. We have exponential growth for $\lambda > 0$; which is condition (5).We will now discuss the roles of the three parameters T_D, i, σ_B:

From (5c) we observe that a percentual increase in σ_B has a stronger effect towards accumulation than a similar increase in i. This may seem counter-intuitive to many, since the focus in this type of discourse is usually the impact of i.

For $\sigma_B = 1$, i.e. all financial income is re-lent, (6) becomes $\dot{D} = iD$, the "classic" equation for accumulation through compound interest, which will then take place for any $i > 0$. An expression of the fascination with – and alarm against – this phenomenon is the table in figure 6 which is a facsimile from

1 Danach ergeben sich vom Jahr 0 bis 1990 bei jährlichem Zinszuschlag folgende ausgewählte Kontostände:

Jahr	Rechnungseinheit DM	Rechnungseinheit kg Gold am 9. 1. 90 18 500,– DM/kg	Rechnungseinheit goldene Erdkugeln (5,973 E + 24 kg)
0	0,01		
95	1,03		
100	1,31		
142	10,20		
189	101,10		
236	1001,55		
296	18708,22	1	
438	19 094 706	1026	
1466	1,16 E+29	6,22 E+24	1
1749	1,148 E+35		1 Mill.
1890	1,116 E+38		1 Mrd.
1990	1,468 E+40	8,026 E+35	134 Mrd.

Heinrich Haußmann: *Der Josephspfennig.* Fürth 1990.

Figure 6: The dramatic dynamics of exponential growth

(Kennedy, 1991). One pfennig (0.01 Deutsche Mark – this was written before the advent of the Euro) deposited in year 0 at 5% interest is by 1990 worth 134 billion massive spheres of gold, each the size of the Earth.

Admittedly, 5% is in real terms a fairly high (real) interest rate, but the table still illustrates the dramatic dynamics of exponential (financial) growth[3].

Another implication of (5) is that cet. par., a large T_D means steeper debt growth. If the loans are perpetuitities ($T_D = \infty$), we have debt growth regardless of the size of σ_B and i, with

$$\dot{D} = i\sigma_B D \tag{7}$$

We get the same result if we assume that all repaid money is lent again, and the lender's costs and consumption are paid out of received interest exclusively, through a share $1 - \sigma_B$ of the interest flow. Then debt growth will occur for any $\sigma_B > 0$, as indicated by (7).

2.3 Accumulation with annuity-type debt service

We now want to check conditions for accumulation (instability) when the DSS is not of the (for simplification purposes) unconventional exponential type as in figure 2, but of the annuity type, shown in

[3] Allegedly also commented like this by Albert Einstein: "the most powerful force in the universe is compound interest." Ironically, this quotation is mostly used today not in the spirit of its critical originator: it is touted to market financial investment.

figure 4. We also in this case choose to ignore the bank time lag subsystem, which is set to unity. The transfer function for the annuity DSS is

$$G_1(s) = \frac{d}{s}(1 - e^{-T_D s}) \tag{8}$$

where d is given by (3), and T_D is the duration of the loan. When we close the loop, we don't get a characteristic polynomial but an irrational expression, due to the term e^{-Ts}. Therefore we cannot check instability via eigenvalues. But since the system is still linear, we may use the Nyquist stability criterion. The loop transfer function $G(s)$ is

$$G(s) = -\sigma_B \frac{d}{s}(1 - e^{-T_D s}) \tag{9}$$

(a minus sign has to be inserted because the criterion is based on the feedback being negative, while the feedback is positive in our case.) The frequency response, given by setting $s = j\omega$ in $G(s)$, is displayed in the form of a polar plot in figure 7. When ω takes on values from $-\infty$ via 0 to ∞, we get a corresponding

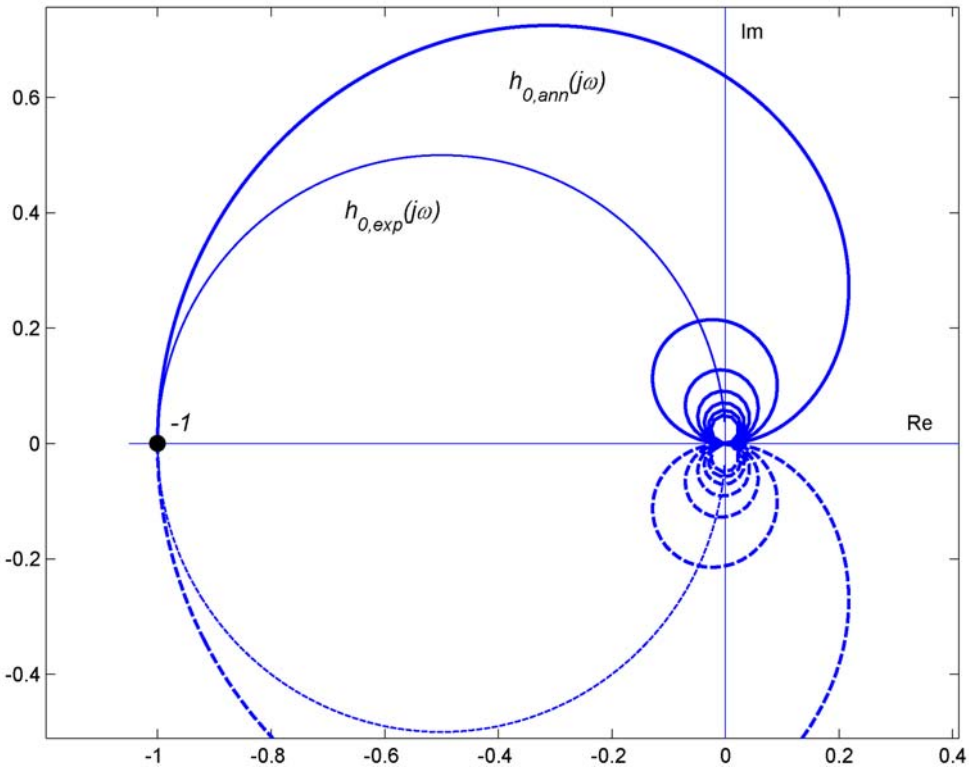

Figure 7: Polar diagrams of $G(s)$ for stability check; annuity and exponential DSS.

closed graph for the frequency response $G(j\omega)$ as displayed in the figure. The dotted half of the graph corresponds to $G(j\omega)$ for $\omega < 0$. $G(s)$ is open-loop stable since the impulse response goes to zero with increasing t. Then the Nyquist criterion simply says that the closed-loop system is stable when the leftmost part of the graph crosses the negative real axis to the right of the point -1.The figure also shows the corresponding graph when the DSS is of the exponential type (where we have already used eigenvalues to check instability). The graph with this DSS is simply a circle, indicated with a thin line. In the figure, the choice of parameters σ_B, i, T_D is such that both graphs go precisely through -1, which means that the two corresponding closed-loop systems are on the border of (in)stability. The chosen parameter values correspond to the two dots in figure 8 below.

While the Nyquist criterion as a general rule can only be applied based on a graph, in this special case we may employ it algebraically. If we consider (9) with $s = j\omega$, we see from angle and absolute value that the leftmost crossing of the negative real axis must take place for $\omega = 0$. We have

$$G(j0) = \lim_{\omega \to 0} \left(-\sigma_B \frac{d}{j\omega}(1 - e^{-j\omega T_D}) \right) = (\text{real}) = -\sigma_B T_D d \tag{10}$$

We substitute (3) for d. The Nyquist criterion, and (10) then gives the condition for financial accumulation:

$$iT_D > \frac{1 - e^{-iT_D}}{\sigma_B} \tag{11}$$

This may be compared to (5c) for the exponential DSS. A better comparison is achieved if we plot borderline stability graphs for both types of DSS, for different sets of parameters σ_B, i, T_D. This is done in figure 8, with i on the x axis, T_D on the y axis, for four different values of σ_B. The graphs for the

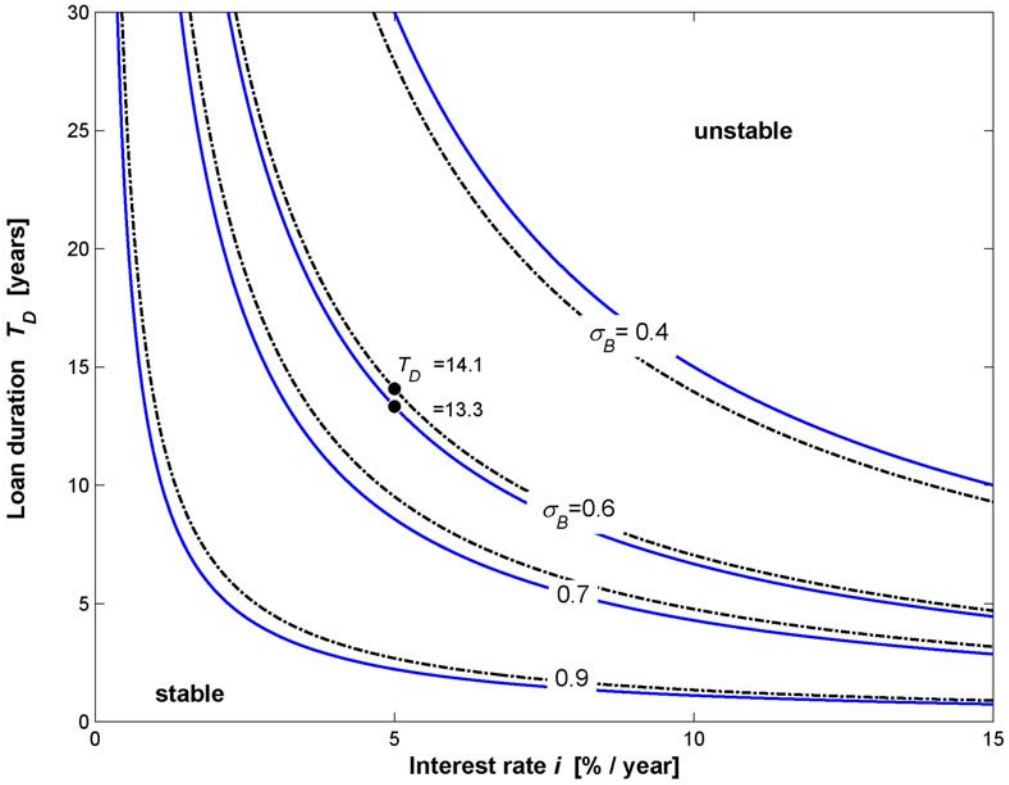

Figure 8: Regions of debt growth ("instability") for values of σ_B, i, T_D

exponential DSS case are solid, while the annuity case graphs are dash-dotted. From the graphs we observe as expected that cet.par., high interest rates or long loan durations give instability (i.e. debt growth, financial accumulation), for both types of DSS. And as already pointed out, an FRC closer to 1 gives debt growth, cet. par. We observe that the graphs for both types of DSS lie fairly close and have similar shapes (all graphs are hyperbolae). This gives support to the notion that the exponential DSS may be used for studying debt growth dynamics instead of the less algebraically tractable annuity DSS.

In the figure, loan duration T_{D2} for the annuity DSS has been adjusted in relation to T_{D1} for the exponential case, following the argument in conjunction with figure 3. In the figure, the T_D on the y axis $= T_{D1}$. By experimenting it was established that $T_{D2} = 1.6T_{D1}$, not $T_{D2} = 2T_{D1}$ as suggested in figure 3, gave the best coincidence for the graphs over a reasonable range of values of σ_B. This adjustment does not, however, invalidate the use of the exponential DSS instead of annuity DSS, since the stability properties of both are so similar.

As an example of how stability information may be extracted from the figure, it is seen that at an interest rate of $5\%/y$ and $\sigma_B = 0.6$, a loan duration $T_{D1} = T_D > 13.3$ will give accumulation when the DSS is exponential, and loan duration $T_{D2} > 1.6T_D = 1.6 \cdot 14.1 = 22.6$ gives accumulation for the annuity DSS case.

2.4 Firms with no income during a start-up period

If we confine ourselves to loans being given to firms (abstracting from household borrowing), the model presupposes that money flows to these firms from day one in the form of demand for consumption and investment goods. Then the firm sector must (be able to) deliver a corresponding flow of products in the opposite direction. How then account for the situation where a firm receives a loan, but for a fair

amount of time will not have any further monetary inflow since it has no products or services to deliver during its build-up phase?

Essentially, the solution is to modify the time profile of debt service, i.e. the impulse response of the debt service subsystem (DSS). If a new loan is extended at $t = 0$, the impulse response of the DSS is now set to zero for an initial period T (perhaps in the order of a year). The firm is exempt from debt service in this period. After $t = T$, debt service starts and follows the same profile(s) as already discussed, but after the original loan has first been amplified by a factor e^{iT} since compound interest must be added before debt service starts. Conditions for accumulation with this modified debt service profile changes somewhat, but the changes are not important for the analysis and quite simple. We will modify the exponential debt service scheme in eq. (1) so that it has the above properties (we could have done the same with the annuity scheme, but it does not make any significant difference for our analysis). The modified transfer function is

$$G_{D'}(s) = e^{iT}e^{-Ts}\frac{1+iT_D}{1+T_Ds} \tag{12}$$

The term e^{iT} accounts for amplifying the debt, and e^{-Ts}accounts for the time delay before debt service starts. Since $G_{D'}(s)$ is irrational due to the term e^{-Ts}, we use the Nyquist criterion to check stability. Following a similar argument as that leading to (9), we now get

$$G(s) = -\sigma_B e^{iT}e^{-Ts}\frac{1+iT_D}{1+T_Ds} \tag{13}$$

Again we may confine ourselves to considering (13) for $s = j\omega$ with $\omega = 0$. We have

$$G(j0) = \left[-\sigma_B e^{iT}e^{-j\omega T}\frac{1+iT_D}{1+T_Dj\omega}\right]_{\omega=0} = (\text{real}) = -\sigma_B e^{iT}\left(1+iT_D\right) \tag{14}$$

The system is unstable (i.e. accumulation occurs) for $-\sigma_B e^{iT}\left(1+iT_D\right) < -1$. This corresponds to conditions for accumulation resembling those in (5):

$$\sigma_B e^{iT} > \frac{1}{(1+iT_D)}, \text{ or equivalently:} \tag{15a}$$

$$i\sigma_B e^{iT} > \frac{1}{T_D}(1-\sigma_B e^{iT}), \text{ or} \tag{15b}$$

$$iT_D > \frac{(1-\sigma_B e^{iT})}{\sigma_B e^{iT}} \tag{15c}$$

As expected, relieving firms of debt service for an initial period with the loan growing correspondingly, moves the system somewhat closer to the instability border for the same set of the three parameters interest, loan duration and banks' financial re-investment coefficient. Comparing (15) to (5), we see that stability-wise, a model with debt relief in an initial period, is equivalent to amplifying the FRC to $\sigma_B = \sigma_B e^{iT}$ in the original model (1).

With debt service relief in an initial period and the extreme special case $\sigma_B e^{iT} > 1 \Longleftrightarrow \sigma_B > e^{-iT}$, conditions (15) tell us that accumulation will always occur.

3 Final remarks

An economic system with lenders recycling financial income as new loans will as a rule be unstable – as warned against since ancient times. For all financial investors (lenders) strive to accumulate. To the degree they succeed, we get increased asset/debt polarisation between lenders and borrowers. Such polarisation occurs since only successful accumulators survive through the market's Darwinian selection process. Thus slow motion debt explosions will be the rule and not the exception. The reason that this is not much recognised or discussed, is probably the time scale for the dynamics involved (several decades), and that the growth path of an exponential function isn't very noticeable until the dramatic late stage.

It also possible that the reason for lack of recognition of the basic accumulation mechanism is – paradoxically – that it is so trivially obvious, if one bothers to think about it. Even the ancient Mesopotamians recognised it. The theory's antique origin, its close relation with religion, and its simplicity all contribute to explain why fringe groups and "crackpots" embrace it. But one should be very careful about dismissing a theory just because it is loved by the fringe. One then has a case of a baby being thrown out with the bathwater. This seems to be the case by parts of the economics profession.

Seen from a control systems perspective however (which ought also to be shared by economists), these runaway long-term dynamics are extremely harmful, and some macroeconomic control mechanism(s) should be implemented.

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Appendices

A Why time-continuous models?

Any model is only an approximation to the real phenomena it tries to represent. Most dynamic economic models are time-discrete. Before the advent of today's sophisticated simulation software, discrete-time models were easier to solve (for example with Excel spreadsheets), which partly explains the discrete-time bias. Another (but erroneous) justification for time-discrete models is that transactions between agents or sectors occur at discrete instances in time, and nothing happens in between. But a time-discrete model presupposes regularly spaced events, while real-world transactions occur with uneven intervals. A precise and elegant way of accounting for such unevenly spaced events is using time-continuous models, but representing the discrete events with delta (impulse) functions: If a unit of money is passed at time $t = t_1$ to an agent or a sector, this mathematically corresponds to an impulse function, commonly symbolised with $\delta(t - t_1)$. This function is a mathematical idealisation: it may be defined as the limit of a rectangular-shaped time function,

$$\delta(t) = \lim_{\varepsilon \longrightarrow 0} \delta_\varepsilon(t), \text{ with } \delta_\varepsilon(t) = \begin{cases} 1/\varepsilon, |t| \leq \varepsilon/2 \\ 0, |t| > \varepsilon/2 \end{cases} . \tag{16}$$

$\delta(t)$ has infinite amplitude and zero duration, but such that its area is unity. $\delta(t)$ is (as approximated by $\delta_\varepsilon(t)$) depicted to the left in figure 9. In an economic model in continuous time, the impulse function allows a correct representation of time-discrete transactions: an amount of money Q passed to a sector or an agent at time t_1 is represented by the function $Q\delta(t - t_1)$. The denomination of this function is money *flow* [$/y], while the area under the function has denomination money *amount* [$]. The *impulse response* $h(t)$ of a unit (in our case an economic agent, a sector or the entire macroeconomic system) is defined as the output signal[4] resulting from one $ input at $t = 0$. The impulse response of a first order linear dynamic system with the input $F_i(t) = \delta(t)$ is

$$F_o(t) = h(t) = \begin{cases} \frac{1}{T} e^{-\frac{t}{T}}, t \geq 0 \\ 0, t < 0 \end{cases} \tag{17}$$

$h(t)$ is shown to the right in figure 9. It is a flow with denomination [$/y]. The area under $h(t)$ is

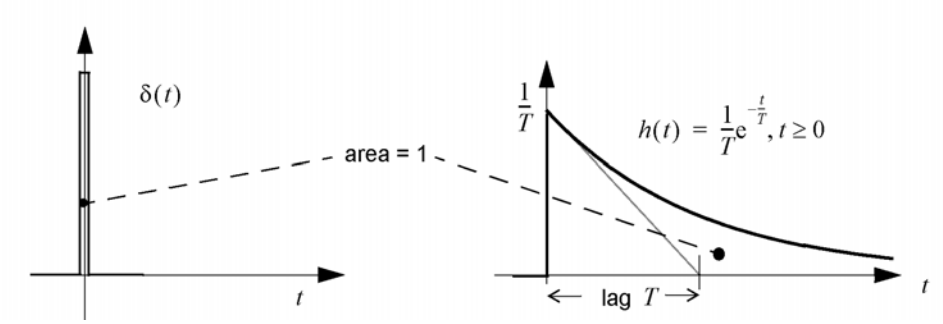

Figure 9: Impulse function (left) and impulse response (right)

unity[5]. This is as expected, since money is neither created nor destroyed when passing by a unit. The mean time lag of $h(t)$ is

$$\int_0^\infty t h(t) dt = \int_0^\infty t \frac{1}{T} e^{-\frac{t}{T}} dt = T \tag{18}$$

(The mean time lag may be estimated by inspection of the graph for $h(t)$, because T is the value of t at the intersection between the tangent of $h(t)$ at $t = 0$ and the time axis, as indicated to the right in figure 9.)

A further argument in favour of choosing the continuous-time framework is that a train of irregularly spaced impulses (which in fact is the *precise* representation of transactions in continuous time) is very

[4] The symbol $h(t)$ is reserved in the control/signals (and) systems literature to signify the output response to an impulse function, as distinct from responses to other input functions.

[5] This property is also expressed by the unit's transfer function having a static gain of unity.

well approximated by a continuous flow when the incidence of transactions is high. This is portrayed in figure 10. When we are working with aggregates of many agents like firms and all households,

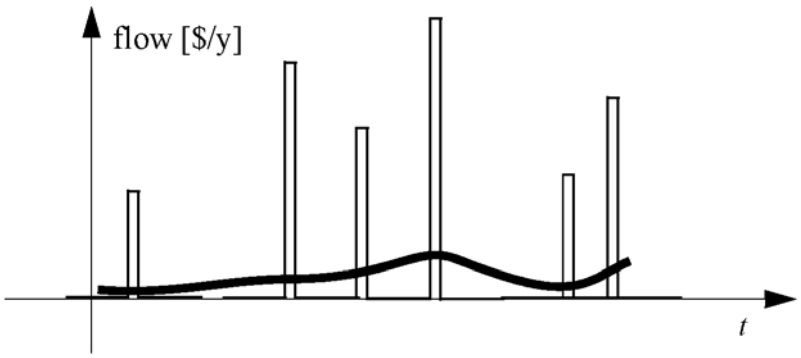

Figure 10: sum of areas under impulses = area under curve [$]

"transaction impulses" between aggregates occur so frequently that continuous flow representation is quite satisfactory. The dynamics of a sector with many units is sluggish related to the incidence of transactions. A subsystem's time constant T expresses this sluggishness (or "inertia"). Interpreted in the frequency domain it is a low pass filter with cutoff frequency $1/T$. Sharp fluctuations in the input will be smoothed out after having passed through. So the output will be similar whether the input is (faithfully) described as a chain of sharp spikes as shown in figure 10 or approximated by the corresponding smooth graph in the same figure.

A further argument for continuous-time representation is that a system may have a large spread in time constants, which is difficult to account for – and also observe by inspection of equations/block diagrams – in time-discrete models. The systems under consideration here exhibit a broad dynamic range from weeks to decades.

Finally, an important advantage with continuous-time representation is that that the response in figure 9 is *dispersed in time*, a property which obviously is present in real-world economic systems: If an amount of money is received by some sector at some instance, the amount will be spread out in time when it is spent. Parts of it will follow a very convoluted path in the sense that it will be used by many agents for transactions *within* the sector, before being paid *out of* the sector[6]. The same holds for money being received by a single agent within a sector at a certain moment; it will not all be spent at once but spread out over time. The first-order continuous time lag model accounts for the dispersed character of the response in a simple, but sufficient manner[7]. The dispersion-in-time property, which holds for all input-output relationships for agents and sectors, *invalidates the approach of analysing monetary circuit dynamics by assuming that these unfold in concluded "periods"*, which is a common assumption in the Post Keynesian/Circuitist literature as mentioned earlier.

B Biblical quotes

> *"When your brother Israelite is reduced to poverty and cannot support himself in the community, you shall assist him as you would an alien and a stranger, and he shall live with you. You shall not charge him interest on a loan, either by deducting it in advance from the capital sum, or by adding it on repayment"* – Leviticus 25:35-36

> *"If you advance money to any poor man amongst my people, you shall not act like a money-lender: you must not exact interest in advance from him"* – Exodus 22:25

> *"You shall not charge interest on anything you lend to a fellow- countryman, money or food or anything else on which interest can be charged. You may charge interest on a loan to foreigner but not on a loan to a fellow countryman..."* – Deuteronomy 23:19-20

[6] The topic of increase in lag for a defined (sub)system due to money circulating within the defined sector/subsystem before leaving it, is comprehensively treated in (Andresen, 1998).

[7] A pioneer in recognising and using this in macroeconomic modeling and simulation, was A.W. Phillips, in a seminal 1954 paper (Phillips, 1954).

"O lord, who may lodge in thy tabernacle? ..... The man .... who does not put his money out to usury ....." – Psalms 15

"He never lends either at discount or at interest. He shuns injustice and deals fairly between man and man" – Ezekiel 18:8-9

"..on the Day of Atonement, You shall send the ram's horn round. You shall send it through all the land to sound a blast, and so you shall hallow the fiftieth year and proclaim liberation in the land for all its inhabitants. You shall make this your year of jubilee. Every man of you shall return to his patrimony, every man to his family.....In this year of the jubilee you shall return, every one of you, to his patrimony... if the man cannot afford to buy back the property, it shall remain in the hands of the purchaser till the year of the jubilee. It shall then revert to the original owner, and he shall return to his patrimony.... When your brother is reduced to poverty and sells himself to you, you shall not use him to work for you as a slave. His status shall be that of a hired man and a stranger lodging with you; he shall work for you until the year of the jubilee. He shall then leave your service, with his children, and go back to his family and to his ancestral property..." – Leviticus 25, excerpts